CHARACTERISTIC FOLIATIONS ON MAXIMALLY REAL SUBMANIFOLDS OF $\mathbb{C}^n$ AND ENVELOPES OF HOLOMORPHY

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ABSTRACT. Let S be an arbitrary real 2-surface, with or without boundary, contained in a hypersurface M of $\mathbb{C}^2$, with S and M of class $\mathcal{C}^{2,\alpha}$, where $0 < \alpha < 1$. If S is totally real except at finitely many complex tangencies which are hyperbolic in the sense of E. Bishop and if the union of separatrices is a tree of curves without cycles, we show that every compact K of S is CR-, $\mathcal{W}$- and L^{p}-removable (Theorem 1.3). Our purely local techniques enable us to formulate substantial generalizations of this statement, for the removability of closed sets in totally real 1-codimensional submanifolds contained in generic submanifolds of CR dimension 1.

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`[With 24 figures]`

§1. INTRODUCTION

In the past fifteen years, remarkable progress has been made towards the understanding of the holomorphic extendability properties of CR functions. At the origin of this development, the most fundamental achievement was the deep discovery, due to the effort of numerous mathematicians, that the so-called *CR orbits* are the adequate underlying objects for the semi-local CR analysis on a general embedded CR manifold. As an independent and now established theory in several complex variables, one may find a precise correspondence between such orbits and progressively attached analytic discs covering a thick part of the envelope of holomorphy of CR manifolds, *cf.* [B], [Trv], [Tr1], [Tu1], [BER], [Tu2], [M1], [J2] and [P3] for a recent synthesis.

Within this framework, it became mathematically accessible to endeavour the general study of removable singularities on embedded CR manifolds $M \subset \mathbb{C}^n$ of arbitrary CR

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dimension and of arbitrary codimension, not necessarily being the boundaries of (strictly) pseudoconvex domains. With respect to their size or "mass", the interesting singularities can be essentially ordered by their codimension in M. For instance, provided it does not perturb the fact that M consists of a single CR orbit, an arbitrary closed subset $C \subset M$ which is of vanishing 2-codimensional Hausdorff content is *always* removable, as is shown in [CS] in the hypersurface case and in [MP3], Theorem 1.1, in arbitrary codimension. Hence one is left to study the removability of singularities of codimension at most two. Since the general problem of characterizing removability seems at the moment to be out of reach (even for M being a hypersurface), it is advisible to focus on geometrically accessible singularities, namely singularities contained in a CR submanifold of M. A complete study of the automatic removability of two-codimensional singularities may be found in Theorem 4 of [MP1]. Having in mind the classical Painlevé problem, we will mainly consider in this paper singularities which are closed sets C contained in a *codimension one* submanifold M^1 of M which is *generic* in $\mathbb{C}^n$.

The known results on singularities of codimension one can be subdivided into two strongly different groups according to the *CR dimension* of M. If $\mathrm{CRdim}\, M \geq 2$, then a generic hypersurface $M^1 \subset M$ is itself a CR manifold of positive CR dimension, and singularities $C \subset M^1$ can be understood on the basis of the interplay between C and the CR orbits of M^1. Deep results in this direction were established when M is a hypersurface of $\mathbb{C}^n$ in [J4], [J5] and then generalized to CR manifolds of arbitrary codimension in [P1]: the geometric condition insuring automatic removability is simply that C does not contain any CR orbit of M^1.

On the other hand, if $\mathrm{CRdim}\, M = 1$ the geometric situation becomes highly different, as a generic hypersurface $M^1 \subset M$ is now (maximal) *totally real*. Fortunately, as a substitute for the CR orbits of M^1, one can consider the so-called *characteristic foliation* of M^1, obtained by integrating the characteristic line field $T^cM|_{M^1} \cap TM^1$. But removability theorems exploiting this concept were only known for hypersurfaces in $\mathbb{C}^2$ and, until very recently, only in the strictly pseudoconvex case. Furthermore, a geometric condition insuring automatic removablitity has not yet been clearly delineated.

Hence, with respect to the current state of the art, there was a two-fold gap about codimension one removable singularities contained in generic submanifolds M of CR dimension one: firstly, to establish a satisfying theory for non-pseudoconvex hypersurfaces in $\mathbb{C}^2$ and secondly, to understand the situation in higher codimension. This second main task was formulated as the first open problem in a list p. 432 of [J5] (*see* also the comments pp. 431–432 about the relative geometric simplicity of the case $\mathrm{CRdim}\, M \geq 2$). *A priori*, it is not clear at all whether the two directions of research are related somehow, but in the present work, we shall fill in this two-fold gap by devising a new semi-local approach which applies uniformly with respect to codimension.

For the detailed discussion of our result we have to introduce some terminology which will be used throughout the article. Let M be a generic submanifold of $\mathbb{C}^n$ and let C be a closed subset of M. Recall from [MP3] that a *wedgelike domain* attached to a generic submanifold $M' \subset \mathbb{C}^n$ is a domain containing a local wedge of edge M' at every point of M'. Our wedgelike domains will always be *nonempty*. Let us define three basic notions of removability. Firstly, we say that C is *CR-removable* if there exists a wedgelike domain $\mathcal{W}$ attached to M to which every continuous CR function $f \in \mathcal{C}^0_{CR}(M \backslash C)$ extends holomorphically. Secondly, as in [MP3], p. 486, we say that C is $\mathcal{W}$*-removable* if for every wedgelike domain $\mathcal{W}_1$ attached to $M \backslash C$, there is a wedgelike domain $\mathcal{W}_2$ attached to M and a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2$ attached to $M \backslash C$ such that for every

holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$. Thirdly, with $\mathsf{p} \in \mathbb{R} \cup \{+\infty\}$ satisfying $\mathsf{p} \geq 1$, we say that C is L^{p}*-removable* if every locally integrable function $f \in L^{\mathsf{p}}_{loc}(M)$ which is CR in the distributional sense on $M \backslash C$ is in fact CR on all of M.

The first notion of removability is a generalization of the kind of removability considered in most of the pioneering papers [CS], [D], [FS], [J1], [KR], [L], [Lu], [St] about removable singularities in boundaries of domains $D \subset\subset \mathbb{C}^n$. We observe that a wedge-like open set attached to a hypersurface M is just a (global) one-sided neighborhood of M, namely a domain ω with $\overline{\omega} \supset M$ such that for every point $p \in M$, the domain ω contains the intersection of a neighborhood of p in $\mathbb{C}^n$ with one side of M. If now a closed set C contained in a $\mathcal{C}^1$-smooth bounded boundary ∂D is CR-removable, then an application of the Hartogs-Bochner theorem shows that CR functions on $\partial D \backslash C$ can be holomorphically extended to D. The second notion of removability is a way to isolate the part of the question related to envelopes of holomorphy. The third notion of removability has the advantage of being completely intrinsic with respect to M and may be relevant in the study of non-embeddable CR manifolds.

To avoid confusion, we state precisely our submanifold notion: Y is a *submanifold of* X if Y and X are equipped with a manifold structure, if there exists an immersion i of Y into X and if the manifold topology of Y and the topology of $i(Y)$ inherited from the topology of X coincide, so that one may identify the submanifold Y with the subset $i(Y) \subset X$. Furthermore, our submanifolds will always be connected.

Let us now enter the discussion of the case $n = 2$. Here we shall denote the submanifold $M^1 \subset M$, which is a surface in $\mathbb{C}^2$, by S. In [B], E. Bishop showed that a two-dimensional surface in $\mathbb{C}^2$ of class at least $\mathcal{C}^2$ having an isolated complex tangency at one of its points p may be represented by a complex equation of the form $w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + \mathrm{o}(|z|^2)$, in terms of local holomorphic coordinates (z, w) vanishing at p, where the real parameter $\lambda \in [0, \infty)$ is a biholomorphic invariant of S. The point p is said to be *elliptic* if $\lambda \in [0, \frac{1}{2})$, *parabolic* if $\lambda = \frac{1}{2}$ and *hyperbolic* if $\lambda \in (\frac{1}{2}, \infty)$. Recall that M is called *globally minimal* if it consists of a single CR orbit (*cf.* [Tr1], [Tr2]; [MP1], pp. 814–815; and [J4], pp. 266–269). Throughout this paper, we shall work in the $\mathcal{C}^{2,\alpha}$-smooth category, where $0 < \alpha < 1$. Our first main new result is as follows.

Theorem 1.1. *Let M be a globally minimal $\mathcal{C}^{2,\alpha}$-smooth hypersurface in $\mathbb{C}^2$ and let $D \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth surface which is*

- **(a)** *$\mathcal{C}^{2,\alpha}$-diffeomorphic to the unit 2-disc of $\mathbb{R}^2$ and*
- **(b)** *totally real outside a discrete subset of isolated complex tangencies which are hyperbolic in the sense of E. Bishop.*

Then every compact subset K of D is CR-, $\mathcal{W}$- and L^{p}-removable.

As a corollary, one obtains a corresponding result about holomorphic extension from $\partial\Omega \backslash K$ for the case that M is the boundary of a relatively compact domain $\Omega \subset \mathbb{C}^2$. Note that $\partial\Omega$ is automatically globally minimal ([J4], Section 2). We will first recall the historical background of Theorem 1.1 and explain afterwards on this basis the main ideas and techniques necessary for the proof.

In 1988, applying a global version of the Kontinuitätssatz, B. Jöricke [J1] established a remarkable theorem: every compact subset of a totally real $\mathcal{C}^2$-smooth 2-disc lying on the boundary of the unit ball in $S^3 = \partial\mathbb{B}_2 \subset \mathbb{C}^2$ is CR-removable. This discovery motivated the work [FS] by F. Forstnerič and E.L. Stout, where it is shown that every $\mathcal{C}^2$-smooth compact 2-disc contained in a strictly pseudoconvex $\mathcal{C}^2$-smooth boundary $\partial\Omega$

contained in a 2-dimensional Stein manifold $\mathcal{M}$ which is totally real except at a finite number of hyperbolic complex tangencies is removable; the proof mainly relies on a previous work by E. Bedford and W. Klingenberg about the hulls of 2-spheres contained in such strictly pseudoconvex boundaries $\Omega \subset \mathcal{M}$, which may be filled by Levi-flat 3-spheres after a generic small perturbation ([BK], Theorem 1). Indirectly, it followed from [J1] and [FS] that such compact totally real 2-discs $D \subset \partial\Omega$ (possibly having finitely many hyperbolic complex tangencies) are $\mathcal{O}(\overline{\Omega})$-convex and in particular polynomially convex if $D = \mathbb{B}_2$ and $\mathcal{M} = \mathbb{C}^2$, thanks to a previous work [St] by E.L. Sout, where it is shown (Theorem II.10) that a compact subset K of a $\mathcal{C}^2$-smooth strictly pseudoconvex boundary $\partial\Omega$ in a Stein manifold is removable *if and only if* K is $\mathcal{O}(\overline{\Omega})$-convex. It is also established in [FS] that a neighborhood of an isolated hyperbolic complex tangency in $\mathbb{C}^2$ is polynomially convex. These papers have been followed by the work [D], where the question of $\mathcal{O}(\overline{\Omega})$-convexity of *arbitrary compact surfaces* S (with or without boundary, not necessarily diffeomorphic to a 2-disc) contained in a $\mathcal{C}^2$-smooth strictly pseudoconvex domain $\Omega \subset \mathbb{C}^2$ is dealt with directly. Using K. Oka's characterization of the envelope of a compact, J. Duval shows that the essential hull $\widehat{K}_{\rm ess} := \overline{\widehat{K}_{\mathcal{O}(\overline{\Omega})} \backslash K}$ must cross every leaf of the characteristic foliation on the totally real part of S and he deduces that a compact 2-disc having only hyperbolic complex tangencies is $\mathcal{O}(\overline{D})$-convex.

All the above proofs heavily rely on strong pseudoconvexity, in contrast to the experience, familiar at least in the case $\mathrm{CRdim}\, M \geq 2$, that removability should depend rather on the structure of CR orbits than on Levi curvature. The first theorem for the non-pseudoconvex situation was established by the second author in [P2]. He proved that every compact subset of a totally real disc embedded in a globally minimal $\mathcal{C}^\infty$-smooth hypersurface in $\mathbb{C}^2$ is always CR-removable. We would like to point out that, seeking theorems without any assumption of pseudoconvexity leads to substantial open problems, because one loses almost all of the strong interweavings between function-theoretic tools and geometric arguments which are valid in the pseudoconvex realm, for instance: Hopf Lemma, plurisubharmonic exhaustions, envelopes of function spaces, local maximum modulus principle, Stein neighborhood basis, *etc.*

To discuss the main elements of our approach, let us briefly explain the geometric setup of the proof of Theorem 1.1. The characteristic foliation has isolated singularities at the hyperbolic points, where it looks like the phase diagram of a saddle point. In particular there are four local separatrices accumulating orthogonally at each hyperbolic point. Hence we can decompose the 2-disc D as a union $D = T_D \cup D_o$, where T_D consists of the union of the hyperbolic points of D together with the separatrices issuing from them, and where $D_o := D \backslash T_D$ is the remaining open submanifold of D, contained in the totally real part of D. By H. Poincaré and I. Bendixson's theory, T_D is a tree of $\mathcal{C}^{2,\alpha}$-smooth curves which contains no subset homeomorphic to the unit circle, *cf.* [D]. Accordingly, we decompose $K := K_{T_D} \cup C_o$, where $K_{T_D} := K \cap T_D$ is a proper closed subset of the tree T_D and where $C_o := K \cap D_o$ is a relatively closed subset of D_o.

The hard part of the proof, which was actually the starting point of the whole paper, will consist in removing the closed subset C_o of the 2-dimensional surface $S := D_o$ lying in $M \backslash T_D$. Thereafter the removal of the remaining part K_{T_D} will be done by means of an investigation of the behaviour of the CR orbits near T_D, close in spirit to our previous methods in [MP1] (*see* Section 12 below for the details).

Let us formulate the first crucial part of the above argument as an independent theorem about the removal of closed subsets contained in a totally real surface S. We point out that now S may have *arbitrary topology*.

Theorem 1.2. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth surface, open or closed, with or without boundary, which is totally real at every point. Let C be a proper closed subset of S and assume that the following topological condition holds*:

$\mathcal{F}_S^c\{C\}$: *For every closed subset $C' \subset C$, there exists a simple $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma : [-1,1] \to S$, whose range is contained in a single leaf of the characteristic foliation $\mathcal{F}_S^c$ (obtained by integrating the characteristic line field $T^cM|_S \cap TS$), with $\gamma(-1) \notin C'$, $\gamma(0) \in C'$ and $\gamma(1) \notin C'$, such that C' lies completely in one closed side of $\gamma[-1,1]$ with respect to the topology of S in a neighborhood of $\gamma[-1,1]$.*

Then C is CR-, $\mathcal{W}$- and L^{p}-removable.

The condition $\mathcal{F}_S^c\{C\}$ is a *common condition* on C and on the characteristic foliation $\mathcal{F}_S^c$, namely on the relative disposition of $\mathcal{F}_S^c$ with respect to C, not only on S; an illustration may be found in FIGURE 2 below. In the strictly pseudoconvex context, this condition appeared implicitly during the course of the proofs given in [D]. Note that the relevance of the characteristic foliation had earlier been discovered in contact geometry, *cf.* [Be], [E]. It is interesting to notice that it re-appears in the situation of Theorem 1.2, where the underlying distribution T^cM is allowed to be very far from contact.

As is known, it follows from a subcase of H. Poincaré and I. Bendixson's theory that if S is diffeomorphic to a real 2-disc or if $S = D_o$ as above, then $\mathcal{F}_S^c\{C\}$ is automatically satisfied for an arbitry compact subset C of S. On the contrary, it may be not satisfied when for instance S is an annulus equipped with a radial foliation together with C containing a continuous closed curve around the hole of S. Crucially, it is elementary to construct an example of such an annulus which is truly nonremovable. Indeed, the small closed curve C which consists of the transversal intersection of a strictly convex boundary ∂D with a complex line close to a boundary point may be enlarged as a thin maximally real strip $S \subset \partial D$ which is diffeomorphic to an annulus; in this setting, C is obviously nonremovable and *the characteristic foliation is everywhere transversal to C*. Consequently, the geometric condition $\mathcal{F}_S^c\{C\}$ is the optimal one insuring automatic removability for all choices of M, S and C. Further examples of closed subsets in surfaces with arbitrary genus equipped with such foliations may be exhibited.

In the proof of Theorem 1.2, after some contraction C' of C, we may assume that no point of C' is locally removable (*see* Sections 2 and 3 below). Then the very assumption $\mathcal{F}_S^c\{C\}$ yields the existence of a characteristic segment $\gamma[-1,1]$, such that C' lies on one side of $\gamma[-1,1]$. Reasoning by contradiction, our aim is to show that there exists at least one special point $p \in C' \cap \gamma(-1,1)$, which is *locally* CR-, $\mathcal{W}$- and L^{p}-removable. The choice of such a point p, achieved in Section 5 below, will be nontrivial.

The strategy for the local removal of p is to construct an analytic disc A such that a segment of its boundary ∂A is attached to S and touches C' in only one point p. Several geometrical assumptions have to be met to ensure that a sufficiently rich family of deformations of A have boundaries disjoint from C', that analytic extension along these discs is possible (i.e. appropriate moment conditions are satisfied), and that the union of these good discs is large enough to give analytic extension to a one-sided neighborhood of M: this is where the (semi)localization and the choice of the special point $p \in C'$ will be key ingredients. Let us explain why localization is crucial.

Working globallly, the second author produced in [P2] a convenient disc by applying the powerful E. Bedford and W. Klingenberg theorem to an appropriate 2-sphere containing a neighborhood of the *entire* singularity C'. This method requires global properties of S like S being a totally real 2-disc, which ensures the existence of a nice Stein neighborhood basis of C'. Already for real discs with isolated hyperbolic points, it is not clear whether this argument can be generalized (however, we would like to mention that recent results of M. Slapar in [Sl] indicate that this could be possible at least if the geometry near the hyperbolic points satisfies some additional assumptions). In the case where M is an arbitrary globally minimal hypersurface, where S has arbitrary topology and has complex tangencies, the reduction to E. Bedford and W. Klingenberg's theorem seems impossible, *cf.* the example of an unknotted nonfillable 2-sphere in $\mathbb{C}^2$ constructed by J.E. Fornæss and D. Ma in [FM]. Also, to the authors' knowledge, the possibility of filling by Levi-flat 3-spheres the (not necessarily generic) 2-spheres lying on a *nonpseudoconvex* hypersurface is a delicate open problem. In addition, for the higher codimensional generalization of Theorem 1.2, the idea of global filling seems to be irrelevant at present times, because no analog of the E. Bedford and W. Klingenberg theorem is known in dimension $n \geq 3$. As we aim to deal with surfaces S having arbitrary topology and to generalize these results in arbitrary codimension, we shall endeavour to firmly *localize the removability arguments*, using only small analytic discs.

Thus, our way to overcome these obstacles is to consider *local* discs A which are only partially attached to S. The delicate point is that we have at the same time (i) to control the geometry of ∂A near $p \in C'$ and (ii) to guarantee that the rest of the boundary stays in the region where holomorphic extension is already known. In fact, (ii) will be incorporated in our very special and tricky choice of $p \in C'$. For (i), we have to sharpen known existence theorems about partially attached analytic discs and to combine it with a careful study of the complex/real geometry of the pair (M, S). Importantly, our construction of such analytic discs is achieved elementarily in a self-contained way. A precise description of the proof in the hypersurface case (only) may be found in Section 2 below. With some substantial extra work, we shall generalize this purely local strategy of proof to higher codimension.

To conclude with the removal of surfaces, let us formulate a more general version of Theorem 1.1, whitout the restricted topological assumption that S be diffeomorphic to a real disc. Applying Theorem 1.2 for the removal of $K \cap (S \backslash T_S)$ and a slight generalization of Theorem 4 (ii) in [MP1] for the removal of $K \cap T_S$ (more precisions will be given in §13 below), we shall obtain the following statement, implying Theorem 1.1 as a direct corollary.

Theorem 1.3. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth totally real surface, open or closed, with or without boundary, which is totally real outside a discrete subset of isolated complex tangencies which are hyperbolic in the sense of E. Bishop. Let T_S be the union of hyperbolic points of S together with all separatrices issued from hyperbolic points and assume that T_S does not contain any subset which is homeomorphich to the unit circle. Let K be a proper compact subset of S and assume that $\mathcal{F}^c_{S \backslash T_S}\{K \cap (S \backslash T_S)\}$ holds.*

Then K is CR-, $\mathcal{W}$- and L^{p}-removable.

As was already emphasized, our main motivation for this work was to *devise a local strategy of proof for Theorems 1.1, 1.2 and 1.3 in order to generalize them to higher codimension*. In fact, we will realize the program sketched above for generic submanifolds of CR dimension 1 and of arbitrary codimension. Thus, let M be a $\mathcal{C}^{2,\alpha}$-smooth globally

minimal generic submanifold of codimension $(n-1)$ in $\mathbb{C}^n$, hence of CR dimension 1, where $n \geq 2$. Let M^1 be a maximally real $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M which is generic in $\mathbb{C}^n$. As in the surface case, M^1 carries a *characteristic foliation* $\mathcal{F}^c_{M^1}$, whose leaves are the integral curves of the line distribution $TM^1 \cap T^cM|_{M^1}$. Of course the assumption that the singularity lies on one side of some characteristic segment is no longer reasonable. We will generalize it as a condition requiring (approximatively speaking) that there be always a characteristic segment which is accessible from the complement of C in M^1 along one direction normal to the characteristic segment.

The generalization of Theorem 1.2, which is our principal result in this paper, is as follows.

Theorem 1.2'. *Let M, M^1, $\mathcal{F}^c_{M^1}$ be as above and let C be a proper closed subset of M. Assume that the following topological condition, meaning that C is not transversal to the characteristic foliation, holds*:

$\mathcal{F}^c_{M^1}\{C\}$: *For every closed subset $C' \subset C$, there exists a simple $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma : [-1,1] \to M^1$ whose range $\gamma[-1,1]$ is contained in a single leaf of the characteristic foliation $\mathcal{F}^c_{M^1}$ with $\gamma(-1) \not\in C'$, $\gamma(0) \in C'$ and $\gamma(1) \not\in C'$, there exists a local $(n-1)$-dimensional transversal $R^1 \subset M^1$ to γ passing through $\gamma(0)$ and there exists a thin elongated open neighborhood V_1 of $\gamma[-1,1]$ in M^1 such that if $\pi_{\mathcal{F}^c_{M^1}} : V_1 \to R^1$ denotes the semi-local projection parallel to the leaves of the characteristic foliation $\mathcal{F}^c_{M^1}$, then $\gamma(0)$ lies on the boundary, relatively to the topology of R^1, of $\pi_{\mathcal{F}^c_{M^1}}(C' \cap V_1)$.*

Then C is CR-, $\mathcal{W}$- and L^{p}-removable.

The condition $\mathcal{F}^c_{M^1}\{C\}$, which is of course independent of the choice of the transversal R^1 and of the thin neighborhood V_1, is illustrated in FIGURE 8 of §5.1 below; clearly, in the case $n = 2$, it means that $C' \cap V_1$ lies completely in one side of $\gamma[-1,1]$, with respect to the topology of M^1, as written in the statement of Theorem 1.2. Applying some of our previous results in this direction ([MP1], [MP3]), we shall provide in the end of Section 13 below some formulations of applications of Theorem 1.2', close to being analogs of Theorem 1.3 in higher codimension.

Importantly, in order to let the geometric condition $\mathcal{F}^c_{M^1}\{C\}$ appear less mysterious and to argue that it provides the adequate generalization of Theorem 1.2 to higher codimension, in the last Section 14 below, we shall describe an example of M, M^1 and C in $\mathbb{C}^3$ violating the condition $\mathcal{F}^c_{M^1}\{C\}$, such that C is transversal to the characteristic foliation and is truly nonremovable. This example will be analogous in some sense to the example of a nonremovable annulus discussed after the statement of Theorem 1.2. Since there is no H. Poincaré and I. Bendixson theorem for foliations of 3-dimensional balls by curves, it will be even possible to insure that M and M^1 are diffeomorphic to real balls of dimension 4 and 3 respectively. We may therefore conclude that Theorem 1.2' provides the desirable answer to the (already cited *supra*) Problem 2.1 raised by B. Jöricke in [J5], p. 432.

To pursue the presentation of our results, let us comment the assumption that M be of codimension $(n-1)$. Geometrically speaking, the study of closed singularities C lying in a one-codimensional generic submanifold M^1 of a generic submanifold $M \subset \mathbb{C}^n$ which is of CR dimension $m \geq 2$ is more simple. Indeed, thanks to the fact that M^1 is of CR dimension $m-1 \geq 1$, there exist local Bishop discs *completely attached to* M^1, and this helps much in describing the envelope of holomorphy of a wedge attached to $M \backslash C$. On the contrary, in the case where M is of CR dimension 1, small analytic

discs attached to a maximally real M^1 are (trivially) inexistent. This is why, in the proof of our main Theorems 1.2 and 1.2', we shall deal only with small analytic discs whose boundary is in part (only) contained in M^1. Such discs are known to exist; we would like to mention that historically speaking, the first construction of discs partially attached to maximally real submanifolds was exhibited by S. Pinchuk in [P], who developed the ideas of E. Bishop [B].

Finally we will test our main Theorem 1.2' in applications. First of all, we clarify its relation to known removability results in CR dimension greater than one. Here the motivation is simply that most questions of CR geometry should be reducible to CR dimension 1 by slicing. It turns out that the main known theorems about removable singularities, due to E. Chirka, E. L. Stout, and B. Jöricke for hypersurfaces, and by the authors in higher codimension ([P1], Theorem 1 about L^{p}-removability; [M2], Theorem 3 about CR- and $\mathcal{W}$- removability) are all a rather direct consequence of Theorem 1.2'. Since these results have not yet been published in complete form, we take the occasion of including them in the present paper, *as a corollary of Theorem 1.2'*, yet devising a new geometric approach.

Theorem 1.4. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 2$ and of codimension $d = n - m \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$ and let $C \subset M^1$ be a proper closed subset of M. Assume that the following condition holds*:

$\mathcal{O}^{CR}_{M^1}\{C\}$: *The closed subset C does not contain any CR orbit of M^1.*

Then C is CR-, $\mathcal{W}$- and L^{p}-removable.

Notice the difference with the case $m = 1$, where the analog of CR orbits would consist of characteristic curves: the condition $\mathcal{F}^c_{M^1}\{C\}$ does *not* say that C should not contain any maximal characteristic curve. In fact, we observe that there cannot exist a uniform removability statement covering both the case $m = 1$ and the case $m \geq 2$, whence Theorem 1.2' is stronger than Theorem 1.4. Indeed, the elementary example of a nonremovable circle in an annulus contained in the boundary of a strictly convex domain of $\mathbb{C}^2$ shows that C may be truly nonremovable whereas it does not contain any characteristic curve. In the strictly pseudoconvex hypersurface setting, it is well known that Hopf's Lemma implies that boundaries of Riemann surfaces contained in C (and also the track on C of its essential hull, *cf.* [D]) should be everywhere transversal to the characteristic foliation. Of course, this implies conversely that C cannot contain such boundaries (unless they are empty) if $\mathcal{F}^c_{M^1}\{C\}$ is satisfied. The reason why $\mathcal{F}^c_{M^1}\{C\}$ implies that C is removable also in the nonpseudoconvex setting and in arbitrary codimension will be appearant later. Finally, we mention that the L^{p}-removability of C in Theorem 1.4 holds more generally with no assumption of global minimality on M, as already noticed in [J5], [P1], [MP1]. However, since the case where M is not globally minimal essentially reduces to the consideration of its CR orbits, which are globally minimal by definition, we shall only deal with globally minimal generic submanifolds M throughout this paper.

As a final comment, we point out that, *because the previously known proofs of Theorem 1.4 were of local type, it is satisfactory to bring in this paper a purely local framework for the treatment of Theorems 1.1, 1.2, 1.3 and 1.2'*.

Our second group of applications concerns the classical edge of the wedge theorem. Typically one considers a maximally real edge E to which an open double wedge $(\mathcal{W}_1, \mathcal{W}_2)$ is attached from opposite directions. One may interprete this configuration as a partial thickening of a generic CR manifold $M \subset E \cup \mathcal{W}_1 \cup \mathcal{W}_2$ containing E as a generic hypersurface. The classical edge of the wedge theorem states that functions which

are continuous on $\mathcal{W}_1 \cup \mathcal{W}_2 \cup E$ and holomorphic in $\mathcal{W}_1 \cup \mathcal{W}_2$ extend holomorphically to a neighborhood of E. Theorem 1.2' implies that it suffices to assume continuity outside a removable singularities of E. This allows us to derive an edge of the wedge theorem for meromorphic extension (Section 13 below).

This paper is divided in two parts: Part I contains the strategy per absurdum for the proof of Theorem 1.2', the construction of what we call a semi-local half-wedge and the choice of a special point to be removed locally. Part II contains the explicit construction of families of half-attached analytic discs, the end of proof of Theorem 1.2' and the proofs of the various applications. The reader will find a more detailed description of the content of the paper in §2.16 below.

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§2. Description of the proof of Theorem 1.2 and organization of the paper

The main part of this paper is devoted to the proof of Theorem 1.2', which will occupy Sections 3, 4, 5, 6, 7, 8 and 9 below. In this preliminary section, we shall summarize the hypersurface version of Theorem 1.2', namely Theorem 1.2. Our goal is to provide a conceptional description of the basic geometric constructions, which should be helpful to read the whole paper. Because precise, complete and rigorous formulations will be developed in the next sections, we allow here the use of a slightly informal language.

2.1. Strategy per absurdum. Let M, S, and C be as in Theorem 1.2. It is essentially known that both the CR- and the L^{p}-removability of C are a (relatively mild) consequence of the $\mathcal{W}$-removability of C (*see* §3.14 and Section 11 below). Thus, we shall describe in this section only the $\mathcal{W}$-removability of C.

First of all, as M is globally minimal, it may be proved that for every closed subset $C' \subset C$, the complement $M \backslash C'$ is also globally minimal (*see* Lemma 3.5 below). As M is of codimension one in $\mathbb{C}^2$, a wedge attached to $M \backslash C$ is simply a connected one-sided neighborhood of $M \backslash C$ in $\mathbb{C}^2$. Let us denote such a one-sided neighborhood by ω_1. The goal is to prove that there exists a one-sided neighborhood ω attached to M to which holomorphic functions in ω_1 extend holomorphically. By the definition of $\mathcal{W}$-removability, this will show that C is $\mathcal{W}$-removable.

Reasoning by contradiction, we shall denote by C_{nr} the smallest *nonremovable* subpart of C. By this we mean that holomorphic functions in ω_1 extend holomorphically to a one-sided neighborhood ω_2 of $M \backslash C_{\text{nr}}$ in $\mathbb{C}^2$ and that C_{nr} is the smallest subset of C such that this extension property holds. If C_{nr} is empty, the conclusion of Theorem 1.2 holds, gratuitously: nothing has to be proved. If C_{nr} is nonempty, to come to an absurd, it suffices to show that at least one point of C_{nr} is *locally removable*. By this, we mean that there exists a local one-sided neighborhood ω_3 of at least one point of C_{nr} such that holomorphic functions in ω_2 extend holomorphically to ω_3. In fact, the choice of such a point will be the most delicate and the most tricky part of the proof.

In order to be in position to apply the continuity principle, we now deform slightly M inside the one-sided neighborhood ω_2, keeping C_{nr} fixed, getting a hypersurface M^d (with d like "*d*eformed") satisfying $M^d \backslash C_{\text{nr}} \subset \omega_2$. We notice that a local one-sided

neighborhood of M^d at one point p of $C_{\rm nr}$ always contains a local one-sided neighborhood of M at p (the reader may draw a figure), so we may well work on M^d instead of working on M (however, the analogous property about wedges over deformed generic submanifolds is untrue in codimension ≥ 2, *see* §3.16 below, where supplementary arguments are needed).

Replacing the notation $C_{\rm nr}$ by the notation C, the notation M^d by the notation M and the notation ω_2 by the notation Ω, we see that Theorem 1.2 is reduced to the following main proposition, whose formulation is essentially analogous to that of Theorem 1.2, except that it suffices to remove at least one special point.

Proposition 2.2. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal hypersurface in $\mathbb{C}^2$, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth surface which is totally real at every point. Let C be a* nonempty *proper closed subset of S and assume that the nontransversality condition $\mathcal{F}_S^c\{C\}$ formulated in Theorem 1.2 holds. Let Ω be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Then there exists a special point $p_{\rm sp} \in C$ and there exists a local one-sided neighborhood $\omega_{p_{\rm sp}}$ of M in $\mathbb{C}^2$ at $p_{\rm sp}$ such that holomorphic functions in Ω extend holomorphically to $\omega_{p_{\rm sp}}$.*

2.3. Holomorphic extension to a half-one-sided neighborhood of M. The choice of the special point $p_{\rm sp}$ will be achieved in two main steps. According to the nontransversality assumption $\mathcal{F}_S^c\{C\}$, there exists a characteristic segment $\gamma : [-1,1] \to S$ with $\gamma(-1) \not\in C$, with $\gamma(0) \in C$ and with $\gamma(1) \not\in C$ such that C lies in one (closed, semi-local) side of γ in S. As γ is a Jordan arc, we may orient S in M along γ, hence we may choose a semi-local open side $(S_\gamma)^+$ of S in M along γ. In the first main step (to be conducted in Section 4 in the context of the general codimensional case Theorem 1.2'), we shall construct what we call a *semi-local half-wedge $\mathcal{HW}_\gamma^+$ attached to $(S_\gamma)^+$ along γ*. By this, we mean the "half part" of a wedge attached to a neighborhood of the characteristic segment γ in M, which yields a wedge attached to the semi-local one-sided neighborhood $(S_\gamma)^+$. For an illustration, *see* FIGURE 6 below, in which one should replace the notation M^1 by the notation S. Such a half-wedge may be interpreted as a wedge attached to a neighborhood of γ in S which is not arbitrary, but should satisfy a further property: locally in a neighborhood of every point of γ, either the half-wedge contains $(S_\gamma)^+$ or one of its two ribs contains $(S_\gamma)^+$, as illustrated in FIGURE 6 below. Importantly also, the cones of this attached half-wedge should vary continuously as we move along γ, *cf.* again FIGURE 6.

The way how we will construct this half-wedge $\mathcal{HW}_\gamma^+$ is as follows. As illustrated in FIGURE 1 just below, we shall first construct a string of analytic discs $Z_{r:s}(\zeta)$, where r is the approximate radius of $Z_{r:s}(\partial\Delta)$, whose boundaries are contained in $(S_\gamma)^+ \subset M$ and which touch the curve γ only at the point $\gamma(s)$, for every $s \in [-1,1]$, namely $Z_{r:s}(1) = \gamma(s)$ and $Z_{r:s}\left(\partial\Delta \backslash \{1\}\right) \subset (S_\gamma)^+$.

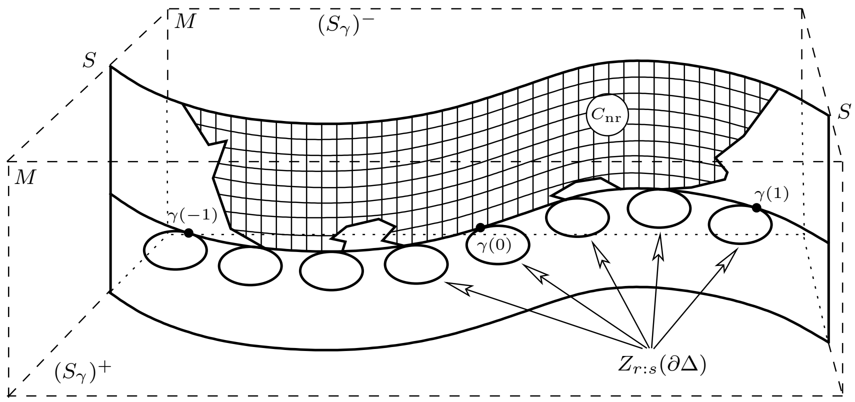

FIGURE 1: STRING OF ANALYTIC DISCS ATTACHED TO $(M_\gamma^1)^+$

From now on, we fix a small radius r_0. By deforming the discs $Z_{r_0:s}(\zeta)$ in Ω near their opposite points $Z_{r_0:s}(-1)$, which lie at a positive distance from the singularity C, we shall construct in Section 4 a family of analytic disc $Z_{r_0,t:s}(\zeta)$, where $t \in \mathbb{R}$ is a small parameter, so that the disc boundaries $Z_{r_0,t:s}(\partial\Delta)$ are pivoting tangentially to S at the point $\gamma(s) \equiv Z_{r_0,t:s}(1)$, which remains fixed as t varies. Precisely, we mean that $\frac{\partial Z_{r_0,t:s}}{\partial\theta}(1) \in T_{\gamma(s)}S$ and that the mapping $t \longmapsto \frac{\partial Z_{r_0,t:s}}{\partial\theta}(1)$ is of rank 1 at $t=0$. This construction and the next ones will be achieved thanks to the solvability Bishop's equation. Furthermore, we may add a small translation parameter $\chi \in \mathbb{R}$, getting a family $Z_{r_0,t,\chi:s}(\zeta)$ with the property that the mapping $(\chi,s) \longmapsto Z_{r_0,t,\chi:s}(1) \in S$ is a diffeomorphism onto a neighborhood of $\gamma([-1,1])$ in S, still with the property that the point $Z_{r_0,t,\chi:s}(1)$ is fixed equal to the point $Z_{r_0,0,\chi:s}(1)$ as t varies. Finally, we may add a small translation parameter $\nu \in \mathbb{R}$ with $\nu > 0$, getting a family $Z_{r_0,t,\chi,\nu:s}(\zeta)$ with $Z_{r_0,t,\chi,0:s}(\zeta) \equiv Z_{r_0,t,\chi:s}(\zeta)$, such that the mapping $(\chi,\nu,s) \longmapsto Z_{r_0,t,\chi,\nu:s}(1)$ is a diffeomorphism onto the semi-local one-sided neighborhood $(S_\gamma)^+$ of S along γ in M, provided $\nu > 0$. Then the semi-local attached half-wedge may be defined as

(2.4)
$$\mathcal{HW}_\gamma^+ := \{Z_{r_0,t,\chi,\nu:s}(\rho) : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ 0 < \nu < \varepsilon, \ 1-\varepsilon < \rho < 1, \ -1 \leq s \leq 1\},$$

for some small $\varepsilon > 0$. In the first main technical step (to be conducted in Section 4 below in the context of Theorem 1.2'), we shall show that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to $\mathcal{HW}_\gamma^+$. To prove Proposition 2.2, we shall find a special point $p_{\rm sp} \in C$ such that there exists a local one-sided neighborhood $\omega_{p_{\rm sp}}$ at $p_{\rm sp}$ such that holomorphic functions in $\Omega \cup \mathcal{HW}_\gamma^+$ extend holomorphically to $\omega_{p_{\rm sp}}$.

2.5. Field of cones on S. We continue the description of the proof of Theorem 1.2 with the full family of analytic discs $Z_{r_0,t,\chi,\nu:s}(\zeta)$. Thanks to a technical application of the implicit function theorem, we can arrange from the beginning that the vectors $\frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1)$ are tangent to S at the point $Z_{r_0,0,\chi,0:s}(1) \in S$ when t varies, for all fixed s. Then by construction, the disc boundaries $Z_{r_0,t,\chi,0:s}(\partial\Delta)$ are pivoting tangentially to S at the point $Z_{r_0,t,\chi,0:s}(1) \equiv Z_{r_0,0,\chi,0:s}(1)$. It follows that when t varies, the oriented half-lines $\mathbb{R}^+ \cdot \frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1)$ describe an open infinite oriented cone in the tangent space to S at the point $Z_{r_0,0,\chi,0:s}(1)$. Consequently, we may define a *field of cones* $p \mapsto \mathsf{C}_p$ as

$$\mathsf{C}_p := \left\{\mathbb{R}^+ \cdot \frac{\partial Z_{r_0,t,\chi,0:s}}{\partial\theta}(1) : \ |t| < \varepsilon\right\}, \tag{2.6}$$

at every point $p = Z_{r_0,0,\chi,0:s}(1) \in S$ of a neighborhood of γ in S. The following figure provides an intuitive illustration. One should think that the small cones are generated when the small discs boundaries of FIGURE 1 pivote tangentially to S.

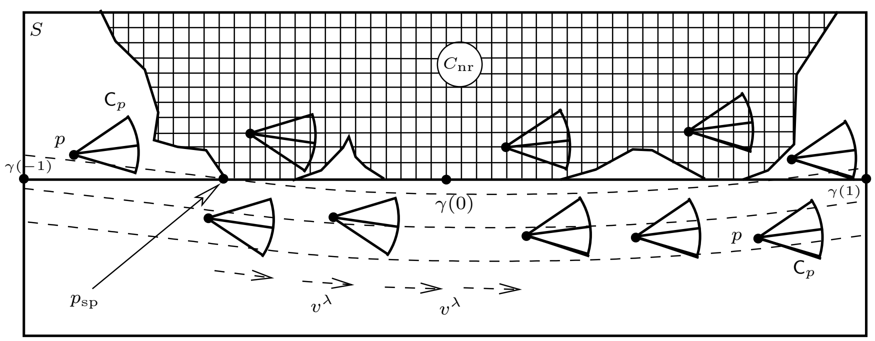

FIGURE 2: FIELD OF CONES ON TS AND CHOICE OF A SPECIAL POINT $p_{\rm sp}$

After having defined this field of cones, we shall *fill* the cones as follows. Remind that a neighborhood of γ in S is foliated by characteristic segments, which are approximatively parallel to γ. In FIGURE 2 above, one should think that the characteristic foliation is horizontal. So there exists a nowhere vanishing vector field $p \mapsto X_p$ defined in a neighborhood of γ whose integral curves are characteristic segments. We define the *filled cone* FC_p by

$$\mathsf{FC}_p := \{\lambda \cdot X_p + (1-\lambda) \cdot v_p : \ 0 \leq \lambda < 1, \ v_p \in \mathsf{C}_p\}. \tag{2.7}$$

Geometrically, we rotate every half-line $\mathbb{R}^+ \cdot v_p$ towards the characteristic half-line $\mathbb{R}^+ \cdot X_p$ and we call the result the *filling* of C_p. In FIGURE 2 above, all the cones C_p coincide with their fillings. Thus we have constructed a field of filled cones $p \longmapsto \mathsf{FC}_p$ over a neighborhood of γ in S.

2.8. Small analytic discs half-attached to S. The next main observation is that small analytic discs which are half-attached to S are essentially contained in the half-wedge $\mathcal{HW}_\gamma^+$, provided that they are approximatively directed by the filled cone FC_p. Let us be more precise. Let $\partial^+\Delta := \{\zeta \in \partial\Delta : \ {\rm Re}\, \zeta \geq 0\}$ denote the *positive half part* of the unit circle. We say that an analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ is *half-attached* to S if $A(\partial^+\Delta)$ is contained in S. Here, A is at least of class $\mathcal{C}^1$ over $\overline{\Delta}$ and holomorphic in Δ. In addition, we shall always assume that our discs A are embeddings of $\overline{\Delta}$ into $\mathbb{C}^2$. We shall say that A is *approximatively straight* (in an informal sense) if $A(\Delta)$ is close in $\mathcal{C}^1$-norm to an open subset of the complex line generated by the complex vector $\frac{\partial A}{\partial \zeta}(1)$. Finally, we say that A is *approximatively directed by the filled cone* FC_p *at* $p = A(1)$, if the vector $\frac{\partial A}{\partial \theta}(1) \in T_pS$ belongs to FC_p. Although this terminology will not be re-employed in the next sections, we may formulate a crucial geometric observation as follows.

Lemma 2.9. *A sufficiently small approximatively straight analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ of class at least $\mathcal{C}^1$ which is half-attached to S and which is approximatively directed by the filled cone FC_p at $p = A(1) \in S$, necessarily satisfies*

$$A\left(\overline{\Delta} \backslash \partial^+\Delta\right) \subset \mathcal{HW}_\gamma^+. \tag{2.10}$$

In the context of the general Theorem 1.2', this property (with more precisions) will be checked in Section 8 below.

2.11. Choice of a special point. In the second main step of the proof (to be conducted in Section 5 in the context of Theorem 1.2'), we shall choose the desired special point $p_{\rm sp}$ of Proposition 2.2 to be removed locally as follows. Since we shall remove $p_{\rm sp}$ by means of half-attached analytic discs (applying the continuity principle), we want to find a special point $p_{\rm sp} \in C$ so that the following two conditions hold true:

- **(i)** There exists a small approximatively straight analytic disc $A : \overline{\Delta} \to \mathbb{C}^2$ with $A(1) = p_{\rm sp}$ which is half-attached to S such that A is approximatively directed by the filled cone $\mathsf{FC}_{p_{\rm sp}}$ (so that the conclusion of Lemma 2.9 above holds true).
- **(ii)** The same disc satisfies $A\,(\partial^+\Delta\backslash\{1\}) \subset S\backslash C$.

In particular, since $M\backslash C$ is contained in Ω, it follows from these two conditions that the blunt disc boundary $A\,(\partial\Delta\backslash\{1\})$ is contained in the open subset $\Omega\cup\mathcal{HW}_\gamma^+$, a property that will be appropriate for the application of the continuity principle, as we shall explain in Section 9 below.

To fulfill conditions **(i)** and **(ii)** above, we first construct a supporting real segment at a special point of the nonempty closed subset $C \subset S$.

Lemma 2.12. *There exists at least one special point $p_{\rm sp} \in C$ arbitrarily close to γ in a neighborhood of which the following two properties hold true*:

- **(i')** *There exists a small $\mathcal{C}^{2,\alpha}$-smooth open segment $H_{p_{\rm sp}} \subset S$ passing through $p_{\rm sp}$ such that an oriented tangent half-line to $H_{p_{\rm sp}}$ at $p_{\rm sp}$ is contained in the filled cone $\mathsf{FC}_{p_{\rm sp}}$, as illustrated in* FIGURE 3 *below.*
- **(ii')** *The same segment is a supporting segment in the following sense*: *locally in a neighborhood of $p_{\rm sp}$, the set $C\backslash\{p_{\rm sp}\}$ is contained in one open side $(H_{p_{\rm sp}})^-$ if $H_{p_{\rm sp}}$ in S, as illustrated in* FIGURE 3 *just below.*

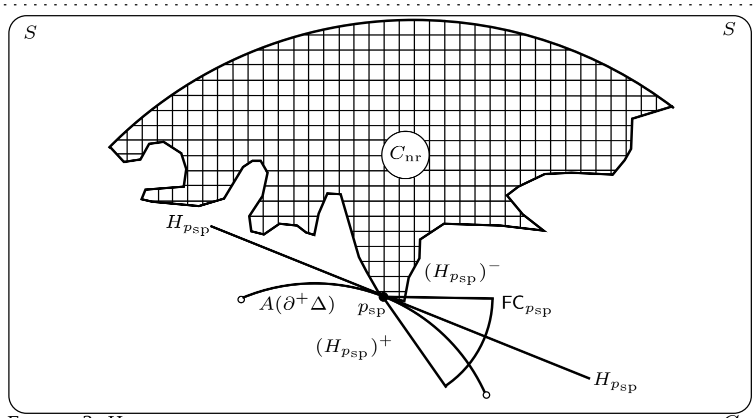

FIGURE 3: HALF-BOUNDARY OF A DISC DIRECTED BY A CONE AND TOUCHING $C_{\rm nr}$

The way how we prove Lemma 2.12 is illustrated intuitively in FIGURE 2 above. For $\lambda \in \mathbb{R}$ with $0 \leq \lambda < 1$ very close to 1, the vector field $p \longmapsto v_p^\lambda := \lambda \cdot X_p + (1-\lambda) \cdot v_p$ is very close to the characteristic vector field $p \mapsto X_p$. By construction, this vector field runs into the filled field of cones $p \mapsto \mathsf{FC}_p$. In FIGURE 2, the integral curves of $p \mapsto v_p^\lambda$ are drawn as dotted lines, which are almost horizontal if λ is very close to 1. If we choose the first dotted integral curve from the lower part of FIGURE 2 which touches C at one special point $p_{\rm sp} \in C$ and if we choose for $H_{p_{\rm sp}}$ a small segment of this first dotted integral curve, we may check that properties **(i)** and **(ii)** are satisfied, modulo some mild

technicalities. A rigorous complete proof of Lemma 2.12 will be provided in Section 5 below.

2.13. Construction of analytic discs half-attached to S. Small analytic discs which are half-attached to a $\mathcal{C}^{2,\alpha}$-smooth maximally real submanifold M^1 of $\mathbb{C}^n$ and which are approximatively straight will be constructed in Section 7 below. For this, we shall use the solution of Bishop's equation with parameters in Hölder spaces, obtained by A. Tumanov in [Tu3] with an optimal loss of regularity. In Section 8, we shall check that it follows from the general constructions of Section 7 that there exists a small analytic disc A half-attached to S with $A(1) = p_{\rm sp}$ whose half boundary is tangent to $H_{p_{\rm sp}}$ at $p_{\rm sp}$ and which satisfies property **(ii)** above, as drawn in FIGURE 3 above. Thus, the two geometric properties **(i')** and **(ii')** satisfied by the real segment $H_{p_{\rm sp}}$ may be realized by the half-boundary of a half-attached analytic disc.

2.14. Translation of half-attached and continuity principle. By means of the results of Section 7, we shall see that we may include the disc $A(\zeta)$ is a parametrized family $A_{x,v}(\zeta)$ of analytic discs half-attached to S, where $x \in \mathbb{R}^2$ and $v \in \mathbb{R}$ are small, so that the mapping $x \mapsto A_{x,0}(1) \in S$ is a local diffeomorphism onto a neighborhood of $p_{\rm sp}$ in S and so that the mapping $v \mapsto \frac{\partial A_{0,v}}{\partial \theta}(1)$ is of rank 1 at $v = 0$. Furthermore, we introduce a new parameter $u \in \mathbb{R}$ in order to "translate" the totally real surface S in M by means of a family of $S_u \subset M$ with $S_0 = S$ and $S_u \subset (S_\gamma)^+$ for $u > 0$. Thanks to the tools developed in Section 7, we deduce that there exists a deformed family of analytic discs $A_{x,v,u}(\zeta)$ which are half-attached to S_u and which satisfies $A_{x,v,0}(\zeta) \equiv A_{x,v}(\zeta)$. In particular, this family covers a local one-sided neighborhood $\omega_{p_{\rm sp}}$ of M at $p_{\rm sp}$ defined by

$$\omega_{p_{\rm sp}} := \{A_{x,v,u}(\rho) : \ |x| < \varepsilon, \ |v| < \varepsilon, \ |u| < \varepsilon, \ 1 - \varepsilon < \rho < 1\}, \tag{2.15}$$

for some $\varepsilon > 0$.

In the third and last main step of the proof (to be conducted in Section 9 below), we shall prove that every disc $A_{x,v,u}(\zeta)$ with $u \neq 0$ is analytically isotopic to a point with the boundary of every disc of the isotopy being contained in $\Omega \cup \mathcal{HW}_\gamma^+$. Thanks to the continuity principle, we shall deduce that every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_\gamma^+\right)$ extends holomorphically to $\omega_{p_{\rm sp}}$ minus a certain thin closed subset $\mathcal{C}_{p_{\rm sp}}$ of $\omega_{p_{\rm sp}}$. Finally, we shall conclude both the proof of Proposition 2.2 and the proof of Theorem 1.2 by checking that the thin closed set $\mathcal{C}_{p_{\rm sp}}$ is in fact removable for holomorphic functions defined in $\omega_{p_{\rm sp}} \backslash \mathcal{C}_{p_{\rm sp}}$.

2.16. Organization of the paper. As was already announced, Sections 3, 4, 5, 6, 7, 8 and 9 below will be entirely devoted to the proof of Theorem 1.2', which will be endeavoured directly in arbitrary codimension, without any further reference to the hypersurface version. Only in Section 3 shall we also consider the beginning of the proof of Theorem 1.4. During the development of the proof of Theorem 1.2', in comparison to the quick description of the proof of Theorem 1.2 achieved just above, we shall unavoidably encounter some supplementary technical complications caused by the codimension being ≥ 2, namely technicalities which are absent in codimension 1. We would like to mention that the crucial geometric argument which enables us to choose the desired special point will be conducted in the central Section 5 below.

Then Section 10 is devoted to summarize three geometrically distinct proofs of Theorem 1.4. In Section 11, we check that both the CR- and the L^{p}-removability of C are a

consequence of the $\mathcal{W}$-removability of C. In Section 12, we provide the proof of Theorem 1.1, of Theorem 1.3 and of further applications. This Section 12 may be read before entering the proof of Theorem 1.2'. Finally, in Section 13, we provide some applications of our removability results to the edge of the wedge theorem for meromorphic functions.

§3. STRATEGY PER ABSURDUM FOR THE PROOFS OF THEOREMS 1.2' AND 1.4

3.1. Preliminary. For the proof of Theorems 1.2', as in [CS], [M2], [MP1], [MP3], [P1], we shall proceed by contradiction. This strategy possesses a considerable advantage: it will not be necessary to control the size of the local subsets of C that are progressively removed, which simplifies substantially the presentation and the understandability of the reasonings. We shall explain how to reduce CR- and $L^{\sf p}$-removability of C to its $\mathcal{W}$-removability. Also, it may be argued that the $\mathcal{W}$-removability of C can be reduced to the simpler case where the functions which we have to extend are even holomorphic in a neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Whereas such a strategy is essentially carried out in detail in previous references (with some variations), we shall for completeness recall the complete reasonings briefly here, in §3.2 and in §3.16 below.

3.2. Global minimality of some complements. For background notions about CR orbits in CR manifolds, we refer the reader to [Su], [J2], [MP1], [J4]. We just recall a few standard facts: if p belongs to a generic submanifold of $\mathbb{C}^n$ of class at least $\mathcal{C}^2$, a point $q \in M$ belongs to the CR orbit $\mathcal{O}_{CR}(M,p)$ if and only if there exists a piecewise smooth curve $\lambda : [0,1] \to M$ with $\lambda(0) = p$, $\lambda(1) = q$ such that $d\lambda(s)/ds \in T^c_{\gamma(s)} M \backslash \{0\}$ at every $s \in [0,1]$ at which λ is differentiable; CR orbits make a partition of M; CR orbits are immersed $\mathcal{C}^{1,\alpha}$-smooth submanifolds of M, according to H.J. Sussmann's Theorem 4.1 in [Su] specialized in the CR category; Every maximal T^cM-integral immersed submanifold of M must contain the CR orbit of each of its points; and finally, a trivial, but often useful fact: if N is a T^cM-integral submanifold of M, namely $T_pN \subset T^c_pM$ for every point $p \in N$, then the local flow of every T^cM-tangent vector field on M stabilizes locally N.

In the two geometric situations of Theorems 1.2' and 1.4, we shall apply the following two Lemmas 3.3 and 3.5 about the CR structure of the complement $M \backslash C'$, where $C' \subset C \subset M^1$ is an arbitrary proper closed subset of C.

Lemma 3.3. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ $(n \geq 2)$ of codimension $d \geq 1$ and of CR dimension $m := n-d \geq 1$, let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M which is generic in $\mathbb{C}^n$ and let C' be an arbitrary proper closed subset of M^1. If either*

- **(1)** *M is of CR dimension $m = 1$ and the condition $\mathcal{F}^c_{M^1}\{C'\}$ of Theorem 1.2' holds; or if*
- **(2)** *M is of CR dimension $m \geq 2$ and the condition $\mathcal{O}^{CR}_{M^1}\{C'\}$ of Theorem 1.4 holds,*

then for each point $p' \in C'$, there exists a piecewise $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma : [0,1] \to M^1$ satisfying $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M \backslash \{0\}$ at every $s \in [0,1]$ at which λ is differentiable, such that $\gamma(0) = p'$ and $\gamma(1)$ does not belong to C'.

Proof. In the case $m = 1$, we proceed by contradiction and we suppose that there exists a point $p' \in C'$ such that all piecewise $\mathcal{C}^{2,\alpha}$-smooth curves $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M \backslash \{0\}$ at every $s \in [0,1]$ at which γ is differentiable, which have origin p' are entirely contained in C'. Notice that since the bundle $TM^1 \cap T^cM|_{M^1}$ is of real dimension one and of class $\mathcal{C}^{1,\alpha}$, such curves γ are in fact $\mathcal{C}^{2,\alpha}$-smooth at every

point. It follows immediately there cannot exist a curve $\gamma : [-1,1] \to M^1$ contained in a single leaf of the characteristic foliation $\mathcal{F}_{M^1}^c$ with $\gamma(0) \in C'$ and $\gamma(-1), \gamma(1) \notin C'$, which contradicts the condition $\mathcal{F}_{M^1}^c\{C'\}$.

In the case $m \geq 2$, by genericity of M^1, the complex tangent bundle T^cM^1, which is of real dimension $(2m-2)$, is a one-codimensional subbundle of the $(2m-1)$-dimensional bundle $TM^1 \cap T^cM|_{M^1}$, namely $T^cM^1 \subset TM^1 \cap T^cM|_{M^1}$. Let $p \in C'$ be an arbitrary point. By the assumption $\mathcal{O}_{M^1}^{CR}\{C'\}$, the CR orbit of p' is not contained in C'. Equivalently, there exists a piecewise $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma : [0,1] \to M^1$ satisfying

$$d\gamma(s)/ds \in T_{\gamma(s)}^c M^1 \backslash \{0\} \subset T_{\gamma(s)} M^1 \cap T_{\gamma(s)}^c M \backslash \{0\} \tag{3.4}$$

at each $s \in [0,1]$ at which γ is differentiable, such that $\gamma(0) = p'$ and $\gamma(1)$ does not belong to C'. Hence the conclusion of Lemma 3.3 is immediately satisfied. This completes the proof. □

As an application, we deduce that under the respective assumptions $\mathcal{F}_{M^1}^c\{C\}$ and $\mathcal{O}_{M^1}^{CR}\{C\}$ of Theorems 1.2' and of Theorem 1.4, the complement $M \backslash C'$ is globally minimal, for every closed subset $C' \subset C$.

Lemma 3.5. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ $(n \geq 2)$ of codimension $d \geq 1$ and of CR dimension $m := n-d \geq 1$, let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M which is generic in $\mathbb{C}^n$ and let C' be an arbitrary* nonempty *proper closed subset of M^1. Assume that for each point $q' \in C'$, there exists a piecewise $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T_{\gamma(s)}^c M \backslash \{0\}$ at every $s \in [0,1]$ at which γ is differentiable, such that $\gamma(0) = q'$ and $\gamma(1)$ does not belong to C'. Then the CR orbit in $M \backslash C'$ of every point $p \in M \backslash C'$ coincides with its CR orbit in M minus C', namely*

$$\mathcal{O}_{CR}(M \backslash C', p) = \mathcal{O}_{CR}(M, p) \backslash C'. \tag{3.6}$$

In particular, as a corollary, if M is globally minimal, then $M \backslash C'$ is also globally minimal.

Proof. Let us first explain the last sentence, which applies to the situations considered in both Theorems 1.2' and 1.4: if $\mathcal{O}_{CR}(M, p) = M$, then by (3.6), $\mathcal{O}_{CR}(M \backslash C', p) \equiv M \backslash C'$, which proves that $M \backslash C'$ is globally minimal.

To establish (3.6), we shall need the following crucial lemma, deserving an illustration: FIGURE 4 below.

Lemma 3.7. *Under the assumptions of Lemma 3.5, for every point $q' \in C' \subset M^1$, there exists a $\mathcal{C}^{1,\alpha}$-smooth locally embedded submanifold $\Omega_{q'}$ of M passing through q' which is transverse to M^1 at q' in M, namely which satisfies $T_{q'}\Omega_{q'} + T_{q'}M^1 = T_{q'}M$, such that*

- **(1)** *$\Omega_{q'}$ is a T^cM-integral submanifold, namely $T_p^cM \subset T_p\Omega_{q'}$, for every point $p \in \Omega_{q'}$.*
- **(2)** *$\Omega_{q'} \backslash C'$ is contained in a single CR orbit of M.*
- **(3)** *$\Omega_{q'}$ is also contained in a single CR orbit of $M \backslash C'$.*

Proof. So, let $q' \in C' \subset M^1$. Since M^1 is generic in $\mathbb{C}^n$, there exists a $\mathcal{C}^{1,\alpha}$-smooth vector field Y defined in a neighborhood of q' which is complex tangent to M but locally transversal to M^1, *cf.* FIGURE 4 just below (for easier readability, we have erased the hatching of C' in a neighborhood of q').

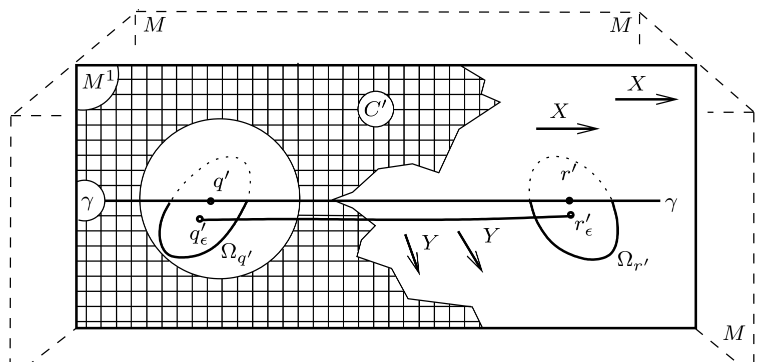

FIGURE 4: CR ORBITS IN THE COMPLEMENT $M\backslash C'$

Following the integral curve of Y issued from q', we can define a point q'_ϵ in an ϵ-neighborhood of q' which does not belong to M^1. By assumption, there exists a piecewise smooth curve $\gamma : [0,1] \to M^1$ with $d\gamma(s)/ds \in T_{\gamma(s)}M^1 \cap T^c_{\gamma(s)}M\backslash\{0\}$ at every $s \in [0,1]$ at which γ is differentiable, such that $\gamma(0) = q'$ and $\gamma(1)$ does not belong to C'. For simplicity, we shall assume that γ consists of a single smooth piece, the case where γ consists of finitely many smooth pieces being treated in a completely similar way. With this assumption (which will simplify slightly the technicalities), it follows that there exists a vector field X defined in a neighborhood of the curve $\gamma([0,1])$ in M which is complex tangent to M and whose restriction to M^1 is a semi-local section of $TM^1 \cap T^cM|_{M^1}$, such that γ is an integral curve of X and such that $\gamma(1) = \exp(X)(q') \in M^1\backslash C'$. In addition, we can assume that the vector field Y is defined in the same neighborhood of $\gamma([0,1])$ in M and everywhere transversal to M^1. If ϵ is sufficiently small, *i.e.* if q'_ϵ is sufficiently close to q', the point $r'_\epsilon := \exp(X)(q'_\epsilon)$ is still very close to M^1. Thus, we can define a new point $r' \in M^1$ to be the unique intersection of the integral of Y issued from r'_ϵ with M^1. By choosing ϵ small enough, the point r'_ϵ will be arbitrarily close to $\gamma(1) \notin C'$, and consequently, we can assume that r' also does not belong to C', as drawn in FIGURE 4 above. Notice that the integral curve of X from q'_ϵ to r'_ϵ is contained in $M\backslash M^1$, since the flow of X stabilizes M^1, whence the two points r'_ϵ and r' belong to the CR orbit $\mathcal{O}_{CR}(M\backslash C', q'_\epsilon)$.

Let $\Omega_{r'}$ denote a small piece of the immersed submanifold $\mathcal{O}_{CR}(M\backslash C', r')$ passing through r'. By the standard properties of CR orbits, we can assume that $\Omega_{r'}$ is an embedded $\mathcal{C}^{1,\alpha}$-smooth submanifold of $M\backslash C'$ of the same CR dimension as $M\backslash C'$ and we can in addition assume that $\Omega_{r'}$ contains r'_ϵ, provided ϵ is small enough. Since Y is a vector field complex tangent to M, the submanifold $\Omega_{r'}$ is necessarily stretched along the flow lines of Y, hence it is transversal to M^1.

We then define the submanifold $\exp(-X)(\Omega_{r'})$, close to the point q' (we shall argue in a while that it passes in fact through q'). Since the flow of X stabilizes M^1, it follows that $\exp(-X)(\Omega_{r'})$ is transversal to M^1 and that $\exp(-X)(\Omega_{r'})$ is divided in two sides by its one-codimensional $\mathcal{C}^{1,\alpha}$-smooth submanifold $M^1 \cap \exp(-X)(\Omega_{r'})$. Furthermore, the flow of X stabilizes the two sides of M^1 in M, semi-locally in a neighborhood of $\gamma([0,1])$, *see* again FIGURE 4 above. Consequently, every integral curve of X issued from every point in $\Omega_{r'}\backslash M^1$ stays in $M\backslash M^1$, hence in $M\backslash C'$ and it follows that the submanifold

$$\exp(-X)(\Omega_{r'})\backslash M^1, \tag{3.8}$$

consisting of two connected pieces, is contained in the single CR orbit $\mathcal{O}_{CR}(M\backslash C', p')$. By the characteristic property of a CR orbit, this means that the two connected pieces of $\exp(-X)(\Omega_{r'})\backslash M^1$ are CR submanifolds of $M\backslash C'$ of the same CR dimension as $M\backslash C'$. Furthermore, since the intersection $M^1 \cap \exp(-X)(\Omega_{r'})$ is one-codimensional, it follows by continuity that *the $\mathcal{C}^{1,\alpha}$-smooth submanifold $\exp(-X)(\Omega_{r'})$ is in fact a CR submanifold of M of the same CR dimension as M*. This proves property **(1)**.

Since q'_ϵ belongs to $\exp(-X)(\Omega_{r'})$ and since the flow of the complex tangent vector field Y necessarily stabilizes the T^cM-integral submanifold $\exp(-X)(\Omega_{r'})$, the point q' which belongs to an integral curve of Y issued from q'_ϵ, must belong to the submanifold $\exp(-X)(\Omega_{r'})$, which we can now denote by $\Omega_{q'}$, as in FIGURE 4 above.

Observe that locally in a neighborhood of q', the integral curves of Y are transversal to M^1 and meet M^1 only at one point. Shrinking if necessary $\Omega_{q'}$ a little bit and using positively or negatively oriented integral curves of Y with origin all points in $\Omega_{q'}\backslash M^1$ lying in both sides, we deduce that $\Omega_{q'}\backslash C'$ is contained in the single CR orbit $\mathcal{O}_{CR}(M\backslash C', r')$, which proves property **(3)**. Using again Y to join points of $C' \cap \Omega_{q'}$, we deduce also that $\Omega_{q'}$ is contained in the single CR orbit $\mathcal{O}_{CR}(M, r')$, which proves property **(2)**.

The proof of Lemma 3.7 is complete. □

We can now prove Lemma 3.5. It suffices to establish that for every two points $p \in M\backslash C'$ and $q \in \mathcal{O}_{CR}(M, p)$ with $q \notin C'$, the point q belongs in fact to the CR orbit of p in $M\backslash C'$, namely q belongs to $\mathcal{O}_{CR}(M\backslash C', p)$.

Since q belongs to the CR orbit of p in M, there exists a piecewise $\mathcal{C}^{2,\alpha}$-smooth curve $\lambda : [0,1] \to M$ with $\lambda(0) = p$, $\lambda(1) = q$ and $d\lambda(s)/ds \in T^c_{\lambda(s)}M\backslash\{0\}$ at every $s \in [0,1]$ at which λ is differentiable. For every s with $0 \leq s \leq 1$, we define a local $\mathcal{C}^{1,\alpha}$-smooth submanifold $\Omega_{\lambda(s)}$ of M passing through $\lambda(s)$ as follows:

- **(1)** If $\lambda(s)$ does not belong to C', choose for $\Omega_{\lambda(s)}$ a piece of the CR orbit of $\lambda(s)$ in $M\backslash C'$.
- **(2)** If $\lambda(s)$ belongs to C', choose for $\Omega_{\lambda(s)}$ the submanifold constructed in Lemma 3.7 above.

By Lemma 3.7, for each s, the complement $\Omega_{\lambda(s)}\backslash C'$ is contained in a single CR orbit of $M\backslash C'$. Since each $\Omega_{\lambda(s)}$ is a T^cM-integral submanifold, for each $s \in [0,1]$, a neighborhood of $\lambda(s)$ in the arc $\lambda([0,1])$ is necessarily contained $\Omega_{\gamma(s)}$. By the Borel-Lebesgue covering lemma, we can therefore find an integer $k \geq 1$ and real numbers

$$0 = s_1 < r_1 < t_1 < s_2 < r_2 < t_2 < \cdots\cdots < s_{k-1} < r_{k-1} < t_{k-1} < s_k = 1, \tag{3.9}$$

such that $\lambda([0,1])$ is covered by $\Omega_{\lambda(0)} \cup \Omega_{\lambda(s_2)} \cup \cdots \cup \Omega_{\lambda(s_{k-1})} \cup \Omega_{\lambda(1)}$ and such that in addition, $\lambda([r_j, t_j]) \subset \Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}$ for $j = 1, \ldots, k-1$.

Lemma 3.10. *The following union minus C'*

$$\big(\Omega_{\lambda(0)} \cup \Omega_{\lambda(s_2)} \cup \cdots\cdots \cup \Omega_{\lambda(s_{k-1})} \cup \Omega_{\lambda(1)}\big)\backslash C' \tag{3.11}$$

is contained in a single CR orbit of $M\backslash C'$.

Proof. It suffices to prove that for every $j = 1, \ldots, k-1$, the union $\big(\Omega_{\lambda(s_j)} \cup \Omega_{\lambda(s_{j+1})}\big)\backslash C'$ minus C' is contained in a single CR orbit of $M\backslash C'$.

Two cases are to be considered. Firstly, assume that $\lambda([r_j, t_j])$ is not contained in C', namely there exists u_j with $r_j \leq u_j \leq t_j$ such that

$$\gamma(u_j) \in \big(\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}\big)\backslash C'. \tag{3.12}$$

Because $\Omega_{\lambda(s_j)}\backslash C'$ and $\Omega_{\lambda(s_{j+1})}\backslash C'$ are both contained in a single CR orbit of $M\backslash C'$, it follows from (3.12) that they are contained in the same CR orbit of $M\backslash C'$, as desired.

Secondly, assume that $\lambda([r_j, t_j])$ is contained in C'. Choose u_j arbitrary with $r_j \leq u_j \leq t_j$. By construction, $\lambda(u_j)$ belongs to $\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}$ and both $\Omega_{\lambda(s_j)}$ and $\Omega_{\lambda(s_{j+1})}$ are T^cM-integral submanifolds of M passing through the point $\lambda(u_j)$. Let Y be a local section of T^cM defined in a neighborhood of $\lambda(u_j)$ which is not tangent to M^1 at $\lambda(u_j)$. On the integral curve of Y issued from $\lambda(u_j)$, we can choose a point $\lambda(u_j)_\epsilon$ arbitrarily close to $\lambda(u_j)$ which does not belong to C'. Since Y is a section of T^cM, it is tangent to both $\Omega_{\lambda(s_j)}$ and $\Omega_{\lambda(s_{j+1})}$, hence we deduce that

$$\gamma(u_j)_\epsilon \in \left(\Omega_{\lambda(s_j)} \cap \Omega_{\lambda(s_{j+1})}\right) \backslash C'. \tag{3.13}$$

Consequently, as in the first case, it follows that $\Omega_{\lambda(s_j)}\backslash C'$ and $\Omega_{\lambda(s_{j+1})}\backslash C'$ are both contained in the same CR orbit of $M\backslash C'$, as desired. This completes the proof of Lemma 3.10. □

Since p and q belong to the set (3.11), we deduce that the points $p = \lambda(0) \in M\backslash C'$ and the point $q = \lambda(1) \in \mathcal{O}_{CR}(M,p)\backslash C'$ belong to the same CR orbit of $M\backslash C'$, as desired. This completes the proof of Lemma 3.5. □

3.14. Reduction of CR- and of L^{p}-removability to $\mathcal{W}$-removability. First of all, we remind that it follows from successive efforts of numerous mathematicians (*cf.* [A], [J2], [M1], [Tr2], [Tu1], [Tu2], [Tu3]) that for every $\mathcal{C}^{2,\alpha}$-smooth globally minimal submanifold M' of $\mathbb{C}^n$, there exists a wedge $\mathcal{W}'$ attached to M to which all continuous CR functions on M extend holomorphically. It follows that the CR-removability of the closed subset $C \subset M^1$ claimed in Theorems 1.2' and 1.4 is an immediate consequence of its $\mathcal{W}$-removability. Based on the construction of analytic discs half-attached to M^1 which will be achieved in Section 7 below, we shall also be able to settle the reduction of L^{p}-removability in the end of the paper, and we formulate a convenient lemma, whose proof is postponed to §11 below.

Lemma 3.15. *Under the assumptions of Theorem 1.2' and of Theorem 1.4, if the closed subset $C \subset M^1$ is $\mathcal{W}$-removable, then it is L^{p}-removable, for all p with $1 \leq \mathsf{p} \leq \infty$.*

3.16. Strategy per absurdum: removal of a single point of the residual non-removable subset. Thus, it suffices to demonstrate that the closed subsets C of Theorems 1.2' and 1.4 are $\mathcal{W}$-removable, *cf.* the definition given in Section 1. Let us fix a wedgelike domain $\mathcal{W}_1$ attached to $M\backslash C$ and remind that all our wedgelike domains are assumed to be nonempty. Our precise goal is to establish that there exists a wedgelike domain $\mathcal{W}_2$ attached to M (including C) and a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2$ attached to $M\backslash C$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$. At first, we need some more definition.

Let C' be an arbitrary closed subset of C. We shall say that $M\backslash C'$ *enjoys the wedge extension property* if there exist a wedgelike domain $\mathcal{W}_2'$ attached to $M\backslash C'$ and a wedgelike subdomain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2'$ attached to $M\backslash C$ such that, for every function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F' \in \mathcal{O}(\mathcal{W}_2')$ which coincides with f in $\mathcal{W}_3$.

The notion of wedge removability can be localized as follows. Let again $C' \subset C$ be arbitrary. We shall say that a point $p' \in C'$ is *locally $\mathcal{W}$-removable with respect to C'* if for every wedgelike domain $\mathcal{W}_1'$ attached to $M\backslash C'$, there exists a neighborhood U' of p' in M, there exists a wedgelike domain $\mathcal{W}_2'$ attached to $(M\backslash C') \cup U'$ and there exists a

wedgelike subdomain $\mathcal{W}_3' \subset \mathcal{W}_1' \cap \mathcal{W}_2'$ attached to $M \backslash C'$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1')$, there exists a holomorphic function $F' \in \mathcal{O}(\mathcal{W}_2')$ which coincides with f in $\mathcal{W}_3'$.

Supppose now that $M \backslash C_1'$ and $M \backslash C_2'$ enjoy the wedge extension property, for some two closed subsets $C_1', C_2' \subset C$. Using Ayrapetian's version of the edge of the wedge theorem ([A], [Tu1], [Tu2]), the two wedgelike domains attached to $M \backslash C_1'$ and to $M \backslash C_2'$ can be glued (after appropriate shrinking) to a wedgelike domain $\mathcal{W}_1$ attached to $M \backslash (C_1' \cap C_2')$ in such a way that $M \backslash (C_1' \cap C_2')$ enjoys the $\mathcal{W}$-extension property. Also, if $M \backslash C'$ enjoys the wedge extension property and if $p' \in C'$ is locally $\mathcal{W}$-removable with respect to C', then again by means of the edge of the wedge theorem, it follows that there exists a neighborhood U' of p' in M such that $(M \backslash C') \cup U'$ enjoys the wedge extension property.

Based on these preliminary remarks, we may define the following set of closed subsets of C:

$$\mathcal{C} := \{ C' \subset C \text{ closed} \ ; \ M \backslash C' \text{ enjoys the } \mathcal{W}\text{-extension property} \}. \tag{3.17}$$

Then the residual set

$$C_{\mathrm{nr}} := \bigcap_{C' \in \mathcal{C}} C' \tag{3.18}$$

is a closed subset of M^1 contained in C. It follows from the above (abstract nonsense) considerations that $M \backslash C_{\mathrm{nr}}$ enjoys the wedge extension property and that no point of C_{nr} is locally $\mathcal{W}$-removable with respect to C_{nr}. Here, we may think that the letters "nr" abbreviate "*n*on-*r*emovable", because by the very definition of C_{nr}, none of its points should be locally $\mathcal{W}$-removable. Notice also that $M \backslash C_{\mathrm{nr}}$ is globally minimal, thanks to Lemma 3.5.

Clearly, to establish Theorem 1.1, it is enough to show that $C_{\mathrm{nr}} = \emptyset$.

We shall argue indirectly (by contradiction) and assume that $C_{\mathrm{nr}} \neq \emptyset$. In order to derive a contradiction, it clearly suffices to show that there exists at least one point $p \in C_{\mathrm{nr}}$ which is in fact locally $\mathcal{W}$-removable with respect to C_{nr}.

At this point, we notice that the main assumptions $\mathcal{F}_{M^1}^c\{C\}$ and $\mathcal{O}_{M^1}^{CR}\{C\}$ of Theorem 1.2' and of Theorem 1.4 imply trivially that for every closed subset C' of C, then the condition $\mathcal{F}_{M^1}^c\{C'\}$ and the condition $\mathcal{O}_{M^1}^{CR}\{C'\}$ also hold true: these two assumptions are obviously stable by passing to closed subsets. In particular, $\mathcal{F}_{M^1}^c\{C_{\mathrm{nr}}\}$ and $\mathcal{O}_{M^1}^{CR}\{C_{\mathrm{nr}}\}$ hold true. Consequently, by following a *per absurdum* strategy, we are led to prove two statements wich are totally similar to Theorem 1.2' and to Theorem 1.4, except that we now have only to establish that *a single point of* C_{nr} is locally $\mathcal{W}$-removable with respect to C_{nr}. This preliminary logical consideration will simplify substantially the whole architecture of the two proofs. Another important advantage of this strategy which will not be appearant until the very end of the two proofs in Sections 9 and 10 below is that we are even allowed to select a special point p_{sp} of C_{nr} by requiring some nice geometric disposition of C_{nr} in a neighborhood of p_{sp} before removing it. Sections 4 and 5 below are devoted to such a selection.

So we are led to show that for every wedgelike domain $\mathcal{W}_1$ attached to $M \backslash C_{\mathrm{nr}}$, there exists a special point $p_{\mathrm{sp}} \in C_{\mathrm{nr}}$, there exists a neighborhood $U_{p_{\mathrm{sp}}}$ of p_{sp} in M, there exists a wedgelike domain $\mathcal{W}_2$ attached to $(M \backslash C_{\mathrm{nr}}) \cup U_{p_{\mathrm{sp}}}$ and there exists a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_2$ attached to $M \backslash C_{\mathrm{nr}}$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$.

A further convenient simplification of the task may be achieved by deforming slightly M inside the wedge $\mathcal{W}_1$ attached to $M \backslash C_{\mathrm{nr}}$. Indeed, by means of a partition of unity,

we may perform arbitrarily small $\mathcal{C}^{2,\alpha}$-smooth deformations M^d of M leaving $C_{\rm nr}$ fixed and moving $M \backslash C_{\rm nr}$ inside the wedgelike domain $\mathcal{W}_1$. Furthermore, we can make M^d to depend on a single small real parameter $d \geq 0$ with $M^0 = M$ and $M^d \backslash C_{\rm nr} \subset \mathcal{W}_1$ for all $d > 0$. Now, *the wedgelike domain $\mathcal{W}_1$ becomes a neighborhood of M^d in $\mathbb{C}^n$*. Let us denote by Ω this neighborhood. After some substantial technical work has been achieved, at the end of the proofs of Theorem 1.2' and 1.4 to be conducted in Sections 9 and 10 below, we shall construct a local wedge $\mathcal{W}_{p_{\rm sp}}^d$ of edge M^d at $p_{\rm sp}$ by means of small Bishop analytic discs glued to M^d, to Ω and to another subset (which we will call a *half-wedge*, *see* Section 4 below) such that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to $\mathcal{W}_{p_{\rm sp}}^d$. Using the stability of Bishop's equation under perturbation, we shall argue in §9.27 below that our constructions are stable under such small deformations, whence in the limit $d \to 0$, the wedges $\mathcal{W}_{p_{\rm sp}}^d$ tend smoothly to a local wedge $\mathcal{W}_{p_{\rm sp}} := \mathcal{W}_{p_{\rm sp}}^0$ of edge a neighborhood $U_{p_{\rm sp}}$ of $p_{\rm sp}$ in $M^0 \equiv M$ (notice that in codimension ≥ 2, a wedge of edge a deformation M^d of M does not contain in general a wedge of edge M, hence such an argument is needed). In addition, we shall derive univalent holomorphic extension to $\mathcal{W}_{p_{\rm sp}}$. Finally, using again the edge of the wedge theorem to fill the space between $\mathcal{W}_1$ and $\mathcal{W}_{p_{\rm sp}}$, possibly after appropriate contractions of these two wedgelike domains, we may construct a wedgelike domain $\mathcal{W}_2$ attached to $(M \backslash C) \cup U_{p_{\rm sp}}$ and a wedgelike domain $\mathcal{W}_3 \subset \mathcal{W}_1 \cap \mathcal{W}_{p_{\rm sp}}$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathcal{O}(\mathcal{W}_1)$, there exists a function $F \in \mathcal{O}(\mathcal{W}_2)$ which coincides with f in $\mathcal{W}_3$. In conclusion, we thus reach the desired contradiction to the definition of $C_{\rm nr}$.

As a summary of the above discussion, we have essentially shown that it suffices to prove Theorems 1.2' and 1.4 with the following two extra simplifying assumptions:

- **1)** Instead of functions which are holomorphic in a wedgelike domain attached to $M \backslash C_{\rm nr}$, we consider functions which are holomorphic in a neighborhood Ω of $M \backslash C_{\rm nr}$ in $\mathbb{C}^{m+n}$.
- **2)** Proceeding by contradiction, we have argued that it suffices to remove at least one point of $C_{\rm nr}$.

Consequently, we may formulate the local statement that remains to prove: after replacing $C_{\rm nr}$ by C and M^d by M, we are led to establish the following main assertion, to which Theorems 1.2' and 1.4 are essentially reduced.

Theorem 3.19. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 1$ and of codimension $d = n - m \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$, and let C be a* nonempty *proper closed subset of M^1.*

- **(i)** *If $m = 1$, assume that the condition $\mathcal{F}_{M^1}^c\{C\}$ holds.*
- **(ii)** *If $m \geq 2$, assume that the condition $\mathcal{O}_{M^1}^{CR}\{C\}$ holds.*

Let Ω be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Then there exist a special point $p_{\rm sp} \in C$, there exists a local wedge $\mathcal{W}_{p_{\rm sp}}$ of edge M at $p_{\rm sp}$ and there exists a subneighborhood $\Omega' \subset \Omega$ of $M \backslash C$ in $\mathbb{C}^n$ with $\mathcal{W}_{p_{\rm sp}} \cap \Omega'$ connected such that for every holomorphic function $f \in \mathcal{O}(\Omega)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{W}_{p_{\rm sp}} \cup \Omega'\right)$ which coincides with f in $\mathcal{W}_{p_{\rm sp}} \cap \Omega'$.

However, we remind the necessity of some supplementary arguments about the stability of our constructions under deformation. The proof of our main Theorem 3.19 will occupy Sections 4, 5, 6, 7, 8, 9 and 10 below and the deformation arguments will appear

lastly in §9.27. From now on, the main question is: *How to choose the special point* $p_{\rm sp}$ *to be removed locally* ?

3.20. Choice of a special point $p \in C$ in the CR dimension $m \geq 2$ case. In the case of CR dimension $m \geq 2$, essentially all points of C can play the role of the special point $p_{\rm sp}$. However, since we want to devise a new proof of Theorem 1.4 which differs from the two proofs given in [J4], [P1] and in [M2], it will be convenient to choose a special point $p_1 \in M^1$ which has the property that locally in a neighborhood of p_1, the singularity $C \subset M^1$ lies behind a generic "wall" H^1 contained in M^1 and of codimension one in M^1. Notice that as $m \geq 2$, the dimension of a two-codimensional submanifold H^1 of M is equal to $2m + d - 2 \geq n$, whence M^1 may perfectly be generic in $\mathbb{C}^n$.

Lemma 3.21. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 2$ and of codimension $d = n-m \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$ and let $C \subset M^1$ be a* nonempty *proper closed subset which does not contain any CR orbit of M^1. Then there exists a point $p_1 \in C$ and a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold $H^1 \subset M^1$ passing through p_1 which is* generic *in $\mathbb{C}^n$ such that $C \backslash \{p_1\}$ lies, in a neighborhood of p_1, in one open side $(H^1)^-$ of H^1 in M^1.*

Proof. The proof is completely similar to the proof of Lemma 2.1 in [MP3], *see* especially FIGURE 1, p. 490. □

With Lemma 3.21 at hand, we can now state a more precise version of case **(ii)** of Theorem 3.19, which will be the main removability proposition.

Proposition 3.22. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 2$ and of codimension $d = n - m \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$, let $p_1 \in M^1$, let $H^1 \subset M^1$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M^1 passing through p_1 which is also generic in $\mathbb{C}^n$ (this is possible, thanks to the assumption $m \geq 2$) and let $(H^1)^-$ denote an open local one-sided neighborhood of H^1 in M^1. Suppose that $C \subset M^1$ is a* nonempty *proper closed subset of M^1 with $p_1 \in C$ which is situated, locally in a neighborhood of p_1, only in one side of H^1, namely $C \subset (H^1)^- \cup \{p_1\}$. Let Ω be a neighborhood of $M \backslash C$ in $\mathbb{C}^n$. Then there exists a local wedge $\mathcal{W}_{p_1}$ of edge M at p_1 with $\Omega \cap \mathcal{W}_{p_1}$ connected (shrinking Ω if necessary) such that for every holomorphic function $f \in \mathcal{O}(\Omega)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{W}_{p_1} \cup \Omega\right)$ with $F|_\Omega = f$.*

In the CR dimension $m = 1$ case, the choice of a special point $p \in C$ is much more delicate and will be done in the next two Sections 4 and 5 below, where the analog of Proposition 3.22 appears as the main removability proposition 5.12. In the case $m = 1$, the submanifold M^1 is of real dimension equal to n and it is not difficult to generalize Lemma 3.21, obtaining a submanifold H^1 which is of real dimension $(n-1)$ and totally real (but not generic) in $\mathbb{C}^n$. However, in general, such a point $p_1 \in C \cap H^1$ is *not locally $\mathcal{W}$-removable* in general. For instance, in the hypersurface case $n = 2$, locally in a neighborhood of p_1, the closed set $C \subset (H^1)^- \cup \{p_1\}$ may coincide with the intersection of M with a local complex curve transverse to M at p_1, hence C is not locally removable. In this (trivial) example, the condition $\mathcal{F}^c\{C\}$ is not satisfied and this justifies a more refined geometrical analysis to chase a suitable special point $p_{\rm sp} \in C$ to be removed locally. This is the main purpose of Sections 4 and 5 below.

§4. CONSTRUCTION OF A SEMI-LOCAL HALF WEDGE

4.1. Preliminary. Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m = 1$ and of codimension $d = n - m \geq 1$ and let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$. Let $\gamma : [-1,1] \to M^1$ be a $\mathcal{C}^{2,\alpha}$-smooth curve, embedding the segment $[-1,1]$ into M. Later, in Section 5 below, we shall exploit the geometric condition formulated in Theorem 2.1' that such a characteristic curve should satisfy, but in the present Section 4, we shall not at all take account of this geometric condition. Our goal is to construct a semi-local half-wedge attached to a one-sided neighborhood of M^1 along γ with the property that holomorphic functions in the neighborhood Ω of $M \backslash C$ in $\mathbb{C}^n$ do extend holomorphically to this half-wedge. First of all, we need to define what we understand by the term "half-wedge". Although all the geometric considerations of this section may be generalized to the CR dimension $m \geq 2$ case with slight modifications, we choose to endeavour the exposition in the case $m = 1$, because our constructions are essentially needed only for this case (however in Section 10 below, since we aim to present a new proof of Theorem 1.4, we shall also use the notion of a local half-wedge, without any characteristic curve γ).

4.2. Three equivalent definitions of attached half-wedges. First of all, we define the notion of a local half-wedge. We shall denote by $\Delta_n(p,\delta)$ the open polydisc centered at $p \in \mathbb{C}^n$ of radius $\delta > 0$. Let $p_1 \in M^1$, and let C_1 be an open infinite cone in the normal space $T_{p_1}\mathbb{C}^n / T_{p_1}M$. Classically, a *local wedge of edge M at p_1* is a set of the form: $\mathcal{W}_{p_1} := \{p + \mathsf{c}_1 : p \in M, \, \mathsf{c}_1 \in \mathsf{C}_1\} \cap \Delta_n(p_1, \delta_1)$, for some $\delta_1 > 0$. Sometimes, we shall use the following terminology: if v_1 is a nonzero vector in $T_{p_1}\mathbb{C}^n / T_{p_1}M$, we shall say that $\mathcal{W}_{p_1}$ is a *local wedge at (p_1, v_1)*. This definition seems to be misleading in the sense that different vectors v_1 seem to yield local wedges with different directions, however, there is a concrete geometric meaning in this definition that should be reminded: the positive half-line $\mathbb{R}^+ \cdot v_1$ generated by the vector v_1 is locally contained in the wedge $\mathcal{W}_{p_1}$.

For us, a *local half-wedge of edge M at p_1* will be a set of the form

$$\mathcal{HW}^+_{p_1} := \left\{ p + \mathsf{c}_1 : p \in U_1 \cap (M^1)^+, \, \mathsf{c}_1 \in \mathsf{C}_1 \right\} \cap \Delta_n(p_1, \delta_1). \tag{4.3}$$

This yields a first definition and we shall delineate two further definitions. Let Δ denote the unit disc in $\mathbb{C}$, let $\partial\Delta$ denote its boundary, the unit circle and let $\overline{\Delta} = \Delta \cup \partial\Delta$ denote its closure. Throughout this paper, we shall denote by $\zeta = \rho e^{i\theta}$ the variable of $\overline{\Delta}$ with $0 \leq \rho \leq 1$ and with $|\theta| \leq \pi$.

In fact, our local half-wedges (to be constructed in this section) will be defined by means of a $\mathcal{C}^{2,\alpha-0}$-smooth $\mathbb{C}^n$-valued (remind $\mathcal{C}^{2,\alpha-0} \equiv \bigcap_{\beta<\alpha} \mathcal{C}^{2,\beta}$) mapping $(t, \chi, \nu, \rho) \longmapsto \mathcal{Z}_{t,\chi,\nu}(\rho)$, which comes from parametrized family of analytic discs of the form $\zeta \mapsto \mathcal{Z}_{t,\chi,\nu}(\zeta)$, where the parameters $t \in \mathbb{R}^{n-1}$, $\chi \in \mathbb{R}^n$, $\nu \in \mathbb{R}$ satisfy $|t| < \varepsilon$, $|\chi| < \varepsilon$, $|\nu| < \varepsilon$ for some small $\varepsilon > 0$, and where $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ is holomorphic with respect to ζ in Δ. This mapping will satisfy the following three properties:

- **(i)** $(\chi, \nu) \mapsto \mathcal{Z}_{0,\chi,\nu}(1)$ is a diffeomorphism onto a neighborhood of p_1 in M, the mapping $\chi \mapsto \mathcal{Z}_{0,\chi,0}(1)$ is a diffeomorphism onto a neighborhood of p_1 in M^1 and $(M^1)^+$ corresponds to $\nu > 0$ in the diffeomorphism $(\chi, \nu) \mapsto \mathcal{Z}_{0,\chi,\nu}(1)$.
- **(ii)** $\mathcal{Z}_{t,0,0}(1) = p_1$ and the half-boundary $\mathcal{Z}_{t,\chi,\nu}\left(\left\{e^{i\theta} : |\theta| \leq \frac{\pi}{2}\right\}\right)$ is contained in M for all t, all χ and all ν.

(iii) The vector $v_1 := \frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1) \in T_{p_1}\mathbb{C}^n$ is nonzero and belongs to $T_{p_1}M^1$. Furthermore, the rank of the $\mathbb{R}^{n-1}$-valued $\mathcal{C}^{1,\alpha-0}$-smooth mapping

$$\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \mathcal{Z}_{t,0,0}}{\partial \theta}(1) \in T_{p_1}M^1 \bmod \big(T_{p_1}M^1 \cap T^c_{p_1}M\big) \cong \mathbb{R}^{n-1} \tag{4.4}$$

is maximal equal to $(n-1)$ at $t=0$.

By holomorphicity of the map $\zeta \mapsto \mathcal{Z}_{t,\chi,\nu}(\zeta)$, we have $\frac{\partial \mathcal{Z}_{t,\chi,\nu}}{\partial \theta}(1) = J \cdot \frac{\partial \mathcal{Z}_{t,\chi,\nu}}{\partial \rho}(1)$, where J denotes the complex structure of $T\mathbb{C}^n$. Consequently, because J induces an isomorphism from $T_{p_1}M/T^c_{p_1}M \to T_{p_1}\mathbb{C}^n/T_{p_1}M$, it follows from property **(iii)** above that the vectors $\frac{\partial \mathcal{Z}_{t,0,0}}{\partial \rho}(1)$ cover an open cone containing Jv_1 in the quotient space $T_{p_1}M/T^c_{p_1}M$, as v varies. Then a *local half-wedge of edge* $(M^1)^+$ *at* p_1 will be a set of the form

$$\mathcal{HW}^+_{p_1} := \{\mathcal{Z}_{t,\chi,\nu}(\rho) \in \mathbb{C}^n : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ 0 < \nu < \varepsilon, \ 1-\varepsilon < \rho < 1\}. \tag{4.5}$$

We notice that a complete local wedge of edge M at p_1 is also associated to such a family $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ and may be defined as $\mathcal{W}_{p_1} := \{\mathcal{Z}_{t,\chi,\nu}(\rho) : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ |\nu| < \varepsilon, \ 1-\varepsilon < \rho < 1\}$.

As may be checked, this second definition of a half-wege is *essentially equivalent* to the first one, in the sense that a half wedge in the sense of the first definition always contains a half-wedge in the sense of the second definition, and vice versa, after appropriate shrinkings of open neighborhoods and cones.

Furthermore, we may distinguish two cases: either the vector $v_1 = \frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ is not complex tangent to M at p_1 or it is complex tangent to M at p_1. In the first case, after possibly shrinking $\varepsilon > 0$, it may be checked that a local half-wedge of edge $(M^1)^+$ coincides with the intersection of a (full) local wedge $\mathcal{W}_{p_1}$ of edge M at p_1 with a one-sided neighborhood $(N^1)^+$ of a local hypersurface N^1 which intersects M locally transversally along M^1 at p_1, as drawn in the left hand side of the following figure, where M is of codimension two.

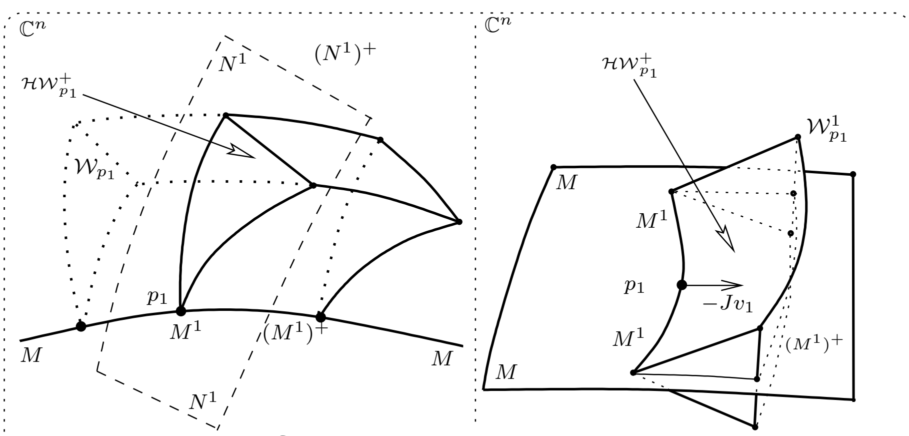

FIGURE 5: GEOMETRIC STRUCTURE OF A LOCAL HALF-WEDGE

In the second case, the vector $v_1 = \frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ is complex tangent to M at p_1, hence belongs to the characteristic direction $T_{p_1}M^1 \cap T^c_{p_1}M$, so the vector $-Jv_1$ which is interiorly tangent to the disc $\mathcal{Z}_{0,0,0}(\Delta)$, is tangent to M at p_1, is not tangent to M^1 at p_1, but points towards $(M^1)^+$ at p_1. It may then be checked that a local half-wedge of edge $(M^1)^+$ coincides with the intersection of a local wedge $\mathcal{W}^1_{p_1}$ of edge M^1 at $(p_1, -Jv_1)$

which contains the side $(M^1)^+$, as drawn in the right hand side of FIGURE 5 above, in which M is of codimension one. This provides the third and the most intuitive definition of the notion of local half-wedge.

Finally, we may define the desired notion of a semi-local attached half-wedge. Let $\gamma : [-1,1] \to M^1$ be an embedded $\mathcal{C}^{2,\alpha}$-smooth segment in M^1. Since the normal bundle to M^1 in M is trivial, we can choose a coherent family of one-sided neighborhoods $(M^1_\gamma)^+$ of M^1 in M along γ. A *half-wedge attached to a one-sided neighborhood* $(M^1_\gamma)^+$ *of* M^1 *along* γ is a domain $\mathcal{HW}^+_\gamma$ which contains a local half-wedge of edge $(M^1)^+$ at $\gamma(s)$ for every $s \in [-1,1]$. Another essentially equivalent definition is to require that we have a family $\mathcal{Z}_{t,\chi,\nu:s}(\rho)$ of mappings smoothly varying with the parameter s such that at each point $\gamma(s) = \mathcal{Z}_{t,\chi,\nu:s}(1)$, the three conditions **(i)**, **(ii)** and **(iii)** introduced above to define a local half-wedge are satisfied. Intuitively speaking, the direction of the cones defining the local half wedge at the point $\gamma(s)$ are smoothly varying with respect to s.

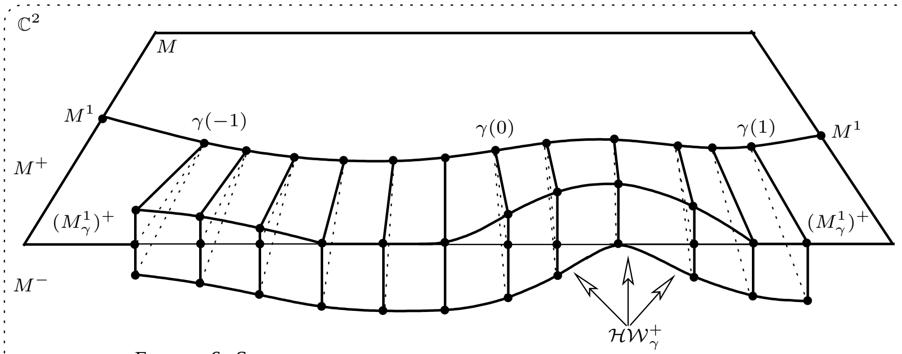

FIGURE 6: SEMI-LOCAL HALF-WEDGE ATTACHED TO A HYPERSURFACE

We can now state the main proposition of this section, which will be of crucial use for the proof of Theorem 3.19 **(i)**. Forgetting for a while the complete content of the geometric condition $\mathcal{F}^c_{M^1}\{C\}$ formulated in the assumptions of Theorem 1.2', which we will analyze thoroughly in Section 5 below, we shall only assume that we are given a characteristic segment $\gamma : [-1,1] \to M^1$ in the following proposition, whose proof is the main goal of this Section 4.

Proposition 4.6. *Let M be a $\mathcal{C}^{2,\alpha}$ globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m = 1$ and of codimension $d = n - m \geq 1$ and let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$. Let $\gamma : [-1,1] \to M^1$ be an arbitrary $\mathcal{C}^{2,\alpha}$-smooth curve. Then there exist a neighborhood V_γ of $\gamma[-1,1]$ in M and a semi-local one-sided neighborhood $(M^1_\gamma)^+$ of M^1 in M along γ which is the intersection of V_γ with a side $(M^1_\gamma)^+$ of M^1 along γ and there exists a semi-local half-wedge $\mathcal{HW}^+_\gamma$ attached to $(M^1_\gamma)^+ \cap V_\gamma$ with $\Omega \cap \mathcal{HW}^+_\gamma$ connected* (*shrinking Ω if necessary*) *such that for every holomorphic function $f \in \mathcal{O}(\Omega)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{HW}^+_\gamma \cup \Omega\right)$ with $F|_\Omega = f$.*

To build $\mathcal{HW}^+_\gamma$, we shall construct families of analytic discs with boundaries in $(M^1_\gamma)^+$. First of all, we need to formulate a special, adapted version of the so-called *approximation theorem* of [BT].

4.7. Local approximation theorem. As noted in [M2], [MP1] and [MP3], when dealing with some natural geometric assumptions on the singularity to be removed – for instance, a two-codimensional singularity $N \subset M$ with $T_pN \supset T_p^cM$ for some points $p \in N$ or metrically thin singularities $E \subset M$ with $H^{2m+d-2}(E) = 0$ – it is impossible to show *a priori* that continuous CR functions on M minus the singularity are approximable by polynomials, which justifies the introduction of deformations and the use of the continuity principle in [M2], [MP1], [MP3]. On the contrary the genericity of the submanifold M^1 containing the singularity C enables us to get an approximation Lemma 4.8 just below, in the spirit of [BT]. Together with the existence of Bishop discs attached to M^1, the validity of this approximation lemma on $M\backslash M^1$ is the second main reason for the relative simplicity of the geometric proofs of Theorem 1.4 provided in [J4], [M2], [P1], in comparison the proof of Theorem 1.2' to be conducted in this paper.

Lemma 4.8. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth globally minimal generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 1$ and of codimension $d = n - m \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$ which divides locally M in two open sides $(M^1)^-$ and $(M^1)^+$ and let $p_1 \in M^1$. Then there exist two neighborhoods U_1 and V_1 of p_1 in M with $V_1 \subset\subset U_1$ such that for every continuous CR function $f \in \mathcal{C}^0_{CR}\left((M^1)^+ \cap U_1\right)$, there exists a sequence of holomorphic polynomials $(P_\nu)_{\nu\in\mathbb{N}}$ wich converges uniformly towards f on $(M^1)^+ \cap V_1$. Of course, the same property holds in the side $(M^1)^-$ instead of $(M^1)^+$.*

Proof. The proof is a slight modification of Proposition 5B in [M2] and we summarize it, taking for granted that the reader is acquainted with the approximation theorem proved in [BT] (*see* also [J5]). Let L_0^1 be a maximally real submanifold passing through p_1 and contained in $M^1 \cap U_1$, for a sufficiently small neighborhood U_1 of p_1 in M^1, possibly to be shrunk later. In coordinates $z = (z_1, \ldots, z_n) = x + iy \in \mathbb{C}^n$ vanishing at p_1, we can assume that the tangent plane to L_0^1 at p_1 identifies with $\mathbb{R}^n = \{y = 0\}$. As the codimensions of L_0^1 in M^1 and in M are equal to $(d-1)$ and to d, we can include L^1 in a d-parameter family of submanifolds L_t^1, where $t \in \mathbb{R}^d$ is small, so that $L_t^1 \cap V_1$ makes a foliation of $M \cap V^1$ by maximally real $\mathcal{C}^{2,\alpha}$-smooth submanifolds, for some neighborhood $V_1 \subset\subset U^1$ of p^1 in M, such that L_t^1 is contained in M^1 for $t = (t_1, \ldots, t_{d-1}, 0)$, *i.e.* for $t_d = 0$, such that $L_t^1 \cap V_1$ is contained in $(M^1)^+$ for $t_d > 0$ and such that $L_t^1 \cap V_1$ is contained in $(M^1)^-$ for $t < 0$. In addition, we can assume that all the L_t^1 coincide with L_0^1 in a neighborhood of ∂U_1.

We shall first treat the case where f is of class $\mathcal{C}^1$. Thus, let f be a $\mathcal{C}^1$-smooth CR function on $\left(M\backslash M^1\right) \cap U_1$, let $\tau \in \mathbb{R}$ with $\tau > 0$, fix $t_0 \in \mathbb{R}^d$ small with $t_{d;0} > 0$, whence $L_{t_0}^1 \cap V^1$ is contained in $(M^1)^+$, let $\widehat{z} \in (M^1)^+ \cap V_1$ be an arbitrary point and consider the following integral which consists of the convolution of f with the Gauss kernel:

$$G_\tau f(\widehat{z}) := \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1 \cap L_{t_0}} e^{-\tau(z-\widehat{z})^2} f(z)\, dz, \tag{4.9}$$

where $(z - \widehat{z})^2 := (z_1 - \widehat{z}_1)^2 + \cdots + (z_n - \widehat{z}_n)^2$ and $dz := dz_1 \wedge \cdots \wedge dz_n$. The point $\widehat{z}$ belongs to some maximally real submanifold $L_{\widehat{t}}$ with $\widehat{t}_d > 0$. We now claim that the value of $G_\tau f(\widehat{z})$ is the same if we replace the integration on the fixed submanifold $U_1 \cap L_{t_0}$ in the integral (4.9) by an integration over $U_1 \cap L_{\widehat{t}}$. This key argument will follows from Stoke's theorem, from the fact that f is CR on $M \cap U_1$ and from the important fact that between L_{t_0} and $L_{\widehat{t}}$, we can construct a $(n + 1)$-dimensional submanifold Σ with

boundary $\partial\Sigma = L_{t_0} - L_{\widehat{t}}$ *which is entirely contained in* $(M^1)^+ \cap U_1$, thanks to the fact that M^1 is generic and of codimension 1; indeed, we may compute

(4.10)
$$\begin{cases} G_\tau f(\widehat{z}) = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1\cap L_{\widehat{t}}} e^{-\tau(z-\widehat{z})^2}\, f(z)\, dz + \left(\frac{\tau}{\pi}\right)^{n/2} \int_{\Sigma} d\left(e^{-\tau(z-\widehat{z})^2}\, f(z)\, dz\right) \\ \qquad\qquad = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{U_1\cap L_{\widehat{t}}} e^{-\tau(z-\widehat{z})^2}\, f(z)\, dz, \end{cases}$$

noticing that the second integral in the right hand side of the first line vanishing, because f and $e^{-(z-\widehat{z})^2}$ being CR and of class at least $\mathcal{C}^1$, one has $d\left(e^{-\tau(z-\widehat{z})^2}\, f(z)\, dz\right) = 0$. This proves the claim.

By analyzing the real and the imaginary part of the phase function $-\tau(z-\widehat{z})^2$ on $L_{\widehat{t}}$, one can show by means of a standard argument (convolution with Gauss' kernel is an approximation of the Dirac mass) that the integral over $U_1 \cap L_{\widehat{t}}$ tends towards $f(\widehat{z})$ as τ tends to ∞, provided that the submanifold $U_1 \cap L_{\widehat{t}}$ is sufficiently close to the real plane $\mathbb{R}^n$ in $\mathcal{C}^1$ norm. Finally, by developing in power series and truncating the exponential in the first expression (4.9) which defines $G_\tau f(\widehat{z})$ and by integrating termwise, one constructs the desired sequence of polynomials $(P_\nu(z))_{\nu\in\mathbb{N}}$.

The case where f is only continuous follows from standard arguments from the theory of distributions. This completes the proof of Lemma 4.8. □

4.11. A family of straightenings. Our main goal in the remainder of this section is to construct a semi-local half-wedge attached to a one-sided neighborhood $(M^1_\gamma)^+$ of M^1 in M along γ which consists of analytic discs attached to $(M^1_\gamma)^+$. First of all, we need a convenient family of normalizations of the local geometries of M and of M^1 along the points $\gamma(s)$ of our characteristic curve γ, for all s with $-1 \leq s \leq 1$.

Let Ω be a thin neighborhood of $\gamma([-1,1])$ in $\mathbb{C}^n$, say a union of polydiscs of fixed radius centered at the points $\gamma(s)$. Then there exists n real valued $\mathcal{C}^{2,\alpha}$-smooth functions $r_1(z,\bar{z}), \dots, r_n(z,\bar{z})$ defined in Ω such that $M\cap\Omega$ is given by the $(n-1)$ Cartesian equations $r_2(z,\bar{z}) = \cdots = r_n(z,\bar{z}) = 0$ and such that moreover, $M^1\cap\Omega$ is given by the n Cartesian equations $r_1(z,\bar{z}) = r_2(z,\bar{z}) = \cdots = r_n(z,\bar{z}) = 0$. We first center the coordinates at $\gamma(s)$ by setting $z' := z - \gamma(s)$. Then the defining functions centered at $z' = 0$ become

$$r_j\left(z' + \gamma(s), \bar{z}' + \overline{\gamma(s)}\right) - r_j\left(\gamma(s), \overline{\gamma(s)}\right) =: r'_j(z', \bar{z}' : s), \tag{4.12}$$

for $j = 1, \dots, n$, and they are parametrized by $s \in [-1,1]$. Now, we drop the primes on coordinates and we denote by $r_j(z,\bar{z} : s)$, $j = 1,\dots,n$, the defining equations for the new M_s and M^1_s, which correspond to the old M and M^1 locally in a neighborhood of $\gamma(s)$. Next, we straighten the tangent planes by using the linear change of coordinates $z' = A_s \cdot z$, where the $n\times n$ matrix A_s is defined by $A_s := 2i\left(\frac{\partial r_j}{\partial z_k}(0,0:s)\right)_{1\leq j,k\leq n}$. Then the defining equations for the two transformed M'_s and for $M^{1'}_s$ are given by

$$r'_j(z', \bar{z}' : s) := r_j\left(A_s^{-1}\cdot z', \overline{A}_s^{-1}\cdot \overline{z}' : s\right), \tag{4.13}$$

and we check immediately that the matrix $\left(\frac{\partial r'_j}{\partial z_k}(0,0:s)\right)_{1\leq j,k\leq n}$ is equal to $2i$ times the $n\times n$ identity matrix, whence $T_0M'_s = \{y'_2 = \cdots = y'_n = 0\}$ and $T_0{M'_s}^1 = \{y'_1 = y'_2 = \cdots = y'_n = 0\}$. It is important to notice that the matrix A_s only depends $\mathcal{C}^{1,\alpha}$-smoothly

with respect to s. Consequently, if we now drop the primes on coordinates, the defining equations for M_s and for M_s^1 are of class $\mathcal{C}^{2,\alpha}$ with respect to $(z, \bar{z})$ and only of class $\mathcal{C}^{1,\alpha}$ with respect to s.

Applying then the $\mathcal{C}^{2,\alpha}$-smooth implicit, we deduce that there exist $(n-1)$ functions $\varphi_j(x, y_1 : s)$, $j = 2, \dots, n$, which are all of class $\mathcal{C}^{2,\alpha}$ with respect to (x, y_1) in a real cube $\mathbb{I}_{n+1}(2\rho_1) := \{(x, y_1) \in \mathbb{R}^n \times \mathbb{R} : \ |x| < 2\rho_1, \ |y_1| < 2\rho_1\}$, for some $\rho_1 > 0$, which are uniformly bounded in $\mathcal{C}^{2,\alpha}$-norm as the parameter s varies in $[-1, 1]$, which are of class $\mathcal{C}^{1,\alpha}$ with respect to s, such that M_s may be represented in the polydisc $\Delta_n(\rho_1)$ by the $(n-1)$ graphed equations

$$y_2 = \varphi_2(x, y_1 : s), \dots\dots, \ y_n = \varphi_n(x, y_1 : s), \tag{4.14}$$

or more concisely $y' = \varphi'(x, y_1 : s)$, if we denote the coordinates $(z_2, \dots, z_n)$ simply by $z' = x' + iy'$. Here, by construction, we have the normalization conditions $\varphi_j(0 : s) = \partial_{x_k}\varphi_j(0 : s) = \partial_{y_1}\varphi_j(0 : s) = 0$, for $j = 2, \dots, n$ and $k = 1, \dots, n$. Sometimes in the sequel, we shall use the notation $\varphi_j(z_1, x' : s)$ instead of $\varphi_j(x, y_1 : s)$. Similarly, again by means of the implicit function theorem, we obtain n functions $h_k(x : s)$, for $k = 1, \dots, n$, which are of class $\mathcal{C}^{2,\alpha}$ in the cube $\mathbb{I}_n(2\rho_1)$ (after possibly shrinking ρ_1) enjoying the same regularity property with respect to s, such that M_s^1 is represented in the polydisc $\Delta_n(\rho_1)$ by the n graphed equations

$$y_1 = h_1(x : s), \ y_2 = h_2(x : s), \ \dots\dots, \ y_n = h_n(x : s). \tag{4.15}$$

In addition, we can assume that

$$h_j(x : s) \equiv \varphi_j\left(x, h_1(x : s) : s\right), \qquad j = 2, \dots, n. \tag{4.16}$$

Here, by construction, we have the normalization conditions $h_k(0 : s) = \partial_{x_l}h_k(0 : s) = 0$ for $k, l = 1, \dots, n$.

In the sequel, we shall denote by $\widehat{z} = \Phi_s(z)$ the final change of coordinates which is centered at $\gamma(s)$ and which straightens simultaneously the tangent planes to M at $\gamma(s)$ and to M^1 at $\gamma(s)$ and we shall denote by M_s and by M_s^1 the transformations of M and of M^1.

Also, we shall remind that the following regularity properties hold for the functions $\varphi_j(x, y_1 : s)$ and $h_k(x : s)$:

- **(a)** For fixed s, they are of class $\mathcal{C}^{2,\alpha}$ with respect to their principal variables, namely excluding the parameter s.
- **(b)** They are of class $\mathcal{C}^{1,\alpha}$ with respect to all their variables, including the parameter s.
- **(c)** Each of their first order partial derivative with respect to one of their principal variables is of class $\mathcal{C}^{1,\alpha}$ with respect to all their variables, including the parameter s.

Indeed, these properties are clearly satisfied for the functions (4.13) and they are inherited after the two applications of the implicit function theorem which yielded the functions $\varphi_j(x, y_1 : s)$ and $h_k(x : s)$.

4.17. Contact of a small "round" analytic disc with M^1. Let $r \in \mathbb{R}$ with $0 \leq r \leq r_1$, where r_1 is small in comparison with ρ_1. Then the "round" analytic disc $\overline{\Delta} \ni \zeta \to \widehat{Z}_{1;r}(\zeta) := ir(1-\zeta) \in \mathbb{C}$ with values in the complex plane equipped with the coordinate $z_1 = x_1 + iy_1$ is centered at the point ir of the y_1-axis, is of radius r and is contained in the open upper half plane $\{z_1 \in \mathbb{C} : y_1 > 0\}$, except its boundary point $\widehat{Z}_{1;r}(1) = 0$. In

addition, the tangent direction $\frac{\partial}{\partial\theta}\widehat{Z}_{1;r}(1) = r$ is directed along the positive x_1-axis, *see* in advance FIGURE 7 below.

As in [Tu2], [Tu3], [MP1], [MP3], we denote by T_1 the Hilbert transform (harmonic conjugate operator) on $\partial\Delta$ vanishing at 1, namely $(T_1 X)(1) = 0$, whence $T_1(T_1(X)) = -X + X(1)$.

By lifting this disc contained in the complex tangent space $T_{p_1}^c M \equiv \mathbb{C}_{z_1} \times \{0\}$, we may define an analytic disc parametrized by r and s which is attached to M of the form

$$\widehat{Z}_{r:s}(\zeta) = \left(ir(1-\zeta), \widehat{Z}'_{r:s}(\zeta) \right) \in \mathbb{C} \times \mathbb{C}^{n-1} \tag{4.18}$$

where the real part $\widehat{X}'_{r:s}(\zeta)$ of $\widehat{Z}'_{r:s}(\zeta)$ satisfies the following Bishop type equation on $\partial\Delta$

$$\widehat{X}'_{r:s}(\zeta) = -\left[T_1 \varphi' \left(\widehat{Z}_{1;r}(\cdot), \widehat{X}'_{r:s}(\cdot) : s \right) \right](\zeta), \qquad \zeta \in \partial\Delta. \tag{4.19}$$

By [Tu1], [Tu3], if r_1 is sufficiently small, there exists a solution which is $\mathcal{C}^{2,\alpha-0}$-smooth with respect to (r,ζ), but only $\mathcal{C}^{1,\alpha-0}$-smooth with respect to (r,ζ,s). Notice that for $r=0$, the disc $\widehat{Z}_{1;0}(e^{i\theta})$ is constant equal to 0 and by uniqueness of the solution of (4.19), it follows that $\widehat{Z}'_{0:s}\left(e^{i\theta}\right) \equiv 0$. It follows trivially that $\partial_\theta \widehat{X}_{0:s}\left(e^{i\theta}\right) \equiv 0$ and that $\partial_\theta\partial_\theta \widehat{X}_{0:s}\left(e^{i\theta}\right) \equiv 0$, which will be used in a while. Notice also that $\widehat{X}_{r:s}(1) = 0$ for all r and all s.

On the other hand, since by assumption, we have $h_1(0:s) = 0$ and $\partial_{x_k} h_1(0:s) = 0$ for $k = 1, \dots, n$, it follows from the chain rule that if we set

$$F(r,\theta:s) := h_1\left(\widehat{X}_{r:s}\left(e^{i\theta}\right) : s \right) \tag{4.20}$$

where θ satisfies $0 \le |\theta| \le \pi$, then the following four equations hold

$$F(0,\theta:s) \equiv 0, \qquad F(r,0:s) \equiv 0, \qquad \partial_\theta F(0,\theta:s) \equiv 0, \qquad \partial_\theta F(r,0:s) \equiv 0. \tag{4.21}$$

We claim that there exists a constant $C > 0$ such that the following five inequalities hold for $0 \le |\theta| \le \pi$, for $0 \le r \le r_1$, for $s \in [-1,1]$ and for $|x| \le \rho_1$:

$$\begin{cases} \left| \widehat{X}_{r:s}\left(e^{i\theta}\right) \right| \le C \cdot r, \\ \left| \partial_\theta \widehat{X}_{r:s}\left(e^{i\theta}\right) \right| \le C \cdot r, \\ \left| \partial_\theta\partial_\theta \widehat{X}_{r:s}\left(e^{i\theta}\right) \right| \le C \cdot r^{\frac{\alpha}{2}}, \\ \sum_{k=1}^{n} |\partial_{x_k} h_1(x)| \le C \cdot |x|, \\ \sum_{k_1,k_2=1}^{n} \left| \partial_{x_{k_1}} \partial_{x_{k_2}} h_1(x) \right| \le C. \end{cases} \tag{4.22}$$

The best constants for each inequality are *a priori* distinct, but we simply take for C the largest one. Indeed, the first, the second and the third inequalities are elementary consequences of the (uniform with respect to s) $\mathcal{C}^{2,\frac{\alpha}{2}}$-smoothness of $\widehat{X}_{r:s}\left(e^{i\theta}\right)$ with respect to (r,θ), and of the normalization conditions (4.21). The fourth and the fifth inequalities are consequences of the $\mathcal{C}^2$-smoothness of h_1 and of its first order normalizations (complete argument may be easily be provided by imitating the reasonings of the elementary Section 6 below).

Computing now the second derivative of $F(r,\theta:s)$ with respect to θ, we obtain
(4.23)

$$\begin{cases}\partial_\theta\partial_\theta F(r,\theta:s)=\sum_{k=1}^{n}\partial_{x_k}h_1\left(\widehat{X}_{r:s}\left(e^{i\theta}\right):s\right)\cdot\partial_\theta\partial_\theta\widehat{X}_{k;r:s}\left(e^{i\theta}\right)+\\ \qquad\qquad+\sum_{k_1,k_2=1}^{n}\partial_{x_{k_1}}\partial_{x_{k_2}}h_1\left(\widehat{X}_{r:s}\left(e^{i\theta}\right)\right)\cdot\partial_\theta\widehat{X}_{k_1;r,s}\left(e^{i\theta}\right)\cdot\partial_\theta\widehat{X}_{k_2;r,s}\left(e^{i\theta}\right),\end{cases}$$

and we may apply the majorations (4.22) to get

$$\text{(4.24)}\qquad\begin{cases}|\partial_\theta\partial_\theta F(r,\theta:s)|\leq C\cdot\left|\widehat{X}_{r:s}(e^{i\theta})\right|\cdot C\cdot r^{\frac{\alpha}{2}}+C\cdot(C\cdot r)^2\\ \qquad\qquad\leq r\cdot C^3\left[r^{\frac{\alpha}{2}}+r^2\right].\end{cases}$$

We can now state and prove a lemma which shows that the disc boundaries $\widehat{Z}_{r:s}(\partial\Delta)$ touches M^1 only at p_1 and lies in $(M^1)^+\cup\{p_1\}$.

Lemma 4.25. *If $r_1\leq\min\left(1,\ \left(\frac{1}{4C^3\pi^2}\right)^{\frac{2}{\alpha}}\right)$, then $\widehat{Z}_{r:s}(\partial\Delta\backslash\{1\})$ is contained in $(M_s^1)^+$ for all r with $0<r\leq r_1$ and all s with $-1\leq s\leq 1$.*

Proof. In the polydisc $\Delta_n(\rho_1)$, the positive half-side $(M_s^1)^+$ in M is represented by the single equation $y_1>h_1(x:s)$, hence we have to check that $\widehat{Y}_{1;r}\left(e^{i\theta}\right)>\left|h_1\left(\widehat{X}_{r:s}\left(e^{i\theta}\right):s\right)\right|$, for all θ with $0<|\theta|\leq\pi$. According to (4.18), the y_1-component $\widehat{Y}_{1;r}\left(e^{i\theta}\right)$ of $\widehat{Z}_{r:s}\left(e^{i\theta}\right)$ is given by $r\left(1-\cos\theta\right)$.

On the first hand, we observe the elementary minoration $r(1-\cos\theta)\geq r\cdot\theta^2\cdot\frac{1}{\pi^2}$, valuable for $0\leq|\theta|\leq\pi$.

On the second hand, taking account of the second and fourth relations (4.21), Taylor's integral formula now yields

$$\text{(4.26)}\qquad F(r,\theta:s)=\int_0^\theta(\theta-\theta')\cdot\partial_\theta\partial_\theta F\left(r,\theta':s\right)\cdot d\theta'.$$

Observing that $r^2\leq r^{\frac{\alpha}{2}}$, since $0<r\leq r_1\leq 1$, and using the majoration (4.24), we may estimate, taking account of the assumption on r_1 written in the statement of the lemma:

$$\text{(4.27)}\qquad|F(r,\theta:s)|\leq r\cdot\frac{\theta^2}{2}\cdot C^3[2r^{\frac{\alpha}{2}}]\leq r\cdot\theta^2\cdot\frac{1}{4\pi^2}.$$

The desired inequality $r(1-\cos\theta)>|F(r,\theta:s)|$ for all $0<|\theta|\leq\pi$ is proved. $\square$

We now fix once for all a radius r_0 with $0<r_0\leq r_1$. In the remainder of the present Section 4, we shall deform the disc $\widehat{Z}_{r_0:s}(\zeta)$ by adding many more parameters. We notice that for all θ with $0\leq|\theta|\leq\frac{\pi}{4}$, we have the trivial minoration $\partial_\theta\partial_\theta\widehat{Y}_{1;r_0}\left(e^{i\theta}\right)=r_0\cos\theta\geq\frac{r_0}{\sqrt{2}}$. Also, by (4.24) and by the inequality on r_1 written in the statement of Lemma 4.25, we deduce $\left|\partial_\theta\partial_\theta h_1\left(\widehat{X}_{r_0:s}\left(e^{i\theta}\right)\right)\right|\leq\frac{r_0}{2\pi^2}$ for all θ with $0\leq|\theta|\leq\pi$. Since we shall need a generalization of Lemma 4.25 in Lemma 4.51 below, let us remember these two interesting inequalities, valid for $0\leq|\theta|\leq\frac{\pi}{4}$:

$$\text{(4.28)}\qquad\begin{cases}\partial_\theta\partial_\theta\widehat{Y}_{1;r_0}\left(e^{i\theta}\right)\geq\dfrac{r_0}{\sqrt{2}},\\ \left|\partial_\theta\partial_\theta h_1\left(\widehat{X}_{r_0:s}\left(e^{i\theta}\right)\right)\right|\leq\dfrac{r_0}{2\pi^2},\end{cases}$$

noticing of course that $\frac{r_0}{2\pi^2}<\frac{r_0}{\sqrt{2}}$.

4.29. Normal deformations of the disc $\widehat{Z}_{r:s}(\zeta)$. So, we fix r_0 small with $0 < r_0 \leq r_1$ and we consider the disc $\widehat{Z}_{r_0:s}(\zeta)$ for $\zeta \in \overline{\Delta}$. Then the point $\widehat{Z}_{r_0:s}(-1)$ belongs to $(M_s^1)^+$ for each s and stays at a positive distance from M_s^1 as s varies in $[-1,1]$. It follows that we can choose a subneighborhood ω_s of $\widehat{Z}_{r_0:s}(-1)$ in $\mathbb{C}^n$ which is contained in Ω and whose diameter is uniformly positive with respect to s.

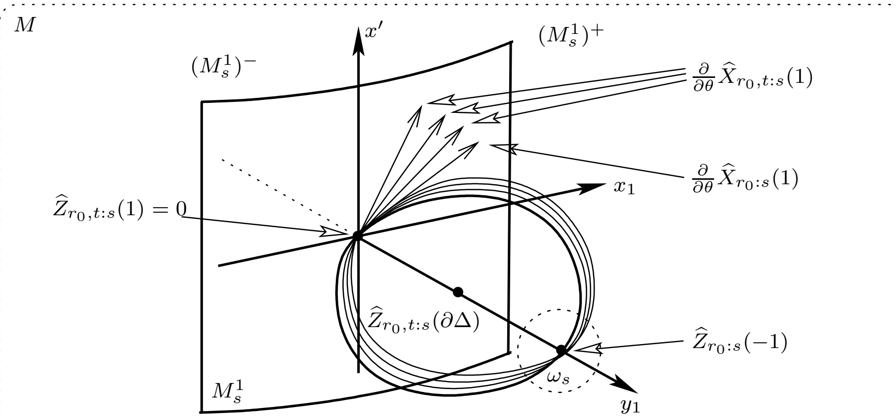

FIGURE 7: NORMAL DEFORMATIONS OF THE ANALYTIC DISC $\widehat{Z}_{r_0:s}(\zeta)$

Following [Tu2] and [MP1], we shall introduce *normal deformations* of the analytic discs $\widehat{Z}_{r_0:s}(\zeta)$ parametrized by s as follows. Let $\kappa : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a $\mathcal{C}^{2,\alpha}$-smooth mapping fixing the origin and satisfying $\partial_{x_k}\kappa_j(0) = \delta_j^k$ for $j,k = 1,\dots,n-1$, where δ_j^k denotes Kronecker's symbol. For $j = 2,\dots,n$, let $\eta_j = \eta_j(z_1, x' : s)$ be a real-valued $\mathcal{C}^{2,\alpha}$-smooth function compactly supported in a neighborhood of the point of $\mathbb{R}^{n+1}$ with coordinates $\left(\widehat{Z}_{1;r_0:s}(-1), \widehat{X}'_{r_0:s}(-1)\right)$ and equal to 1 at this point. We then define the $\mathcal{C}^{2,\alpha}$-smooth deformed generic submanifold $M_{s,t}$ of equations

$$\begin{cases} y' &= \varphi'(z_1, x' : s) + \kappa(t) \cdot \eta'(z_1, x' : s) \\ &=: \Phi'(z_1, x', t : s). \end{cases} \tag{4.30}$$

Notice that $M_{s,0} \equiv M_s$ and that $M_{s,t}$ coincides with M_s in a small neighborhood of the origin, for all t. If $\mu = \mu\left(e^{i\theta} : s\right)$ is a real-valued nonnegative $\mathcal{C}^{2,\alpha}$-smooth function defined for $e^{i\theta} \in \partial\Delta$ and for $s \in [-1,1]$ whose support is concentrated near the segment $\{-1\} \times [-1,1]$, then applying the existence Theorem 1.2 of [Tu3], for each fixed $s \in [-1,1]$, we deduce the existence of a $\mathcal{C}^{2,\alpha-0}$-smooth solution of the Bishop type equation

$$\widehat{X}'_{r_0,r:s}\left(e^{i\theta}\right) = -\left[T_1\Phi'\left(\widehat{Z}_{1;r_0:s}(\cdot), \widehat{X}'_{r_0,t:s}(\cdot), t\mu(\cdot : s) : s\right)\right]\left(e^{i\theta}\right), \tag{4.31}$$

which enable us to construct a deformed family of analytic disc

$$\widehat{Z}_{r_0,t:s}\left(e^{i\theta}\right) := \left(\widehat{Z}_{1;r_0:s}(e^{i\theta}), \widehat{X}'_{r_0,t:s}(e^{i\theta}) + iT_1\left[\widehat{X}'_{r_0,t:s}(\cdot)\right]\left(e^{i\theta}\right)\right) \tag{4.32}$$

whose boundaries are contained in $M \cup \omega_s$, by construction. By an inspection of Theorem 1.2 in [Tu3], taking account of the regularity properties **(a)**, **(b)** and **(c)** stated after (4.16), one can show that the general solution $\widehat{Z}_{r_0,t:s}(\zeta)$ enjoys regularity properties which are completely similar:

(a) For fixed s, it is of class $\mathcal{C}^{2,\alpha-0}$ with respect to (t,ζ).
(b) It is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all the variables (t,ζ,s).
(c) Each of its first order partial derivative with respect to the principal variables (t,ζ) is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all the variables (t,ζ,s).

Since the solution is $\mathcal{C}^{1,\alpha-0}$-smooth with respect to s, it crucially follows that the vector

$$v_{1:s} := -\frac{\partial \widehat{Z}_{r_0,t:s}}{\partial \rho}(1), \tag{4.33}$$

which points inside the analytic disc, varies continuously with respect to s. The following key proposition may be established (up to a change of notations) just by reproducing the proof of Lemma 2.7 in [MP1], taking account of the uniformity of all differentiations with respect to the curve parameter s.

Lemma 4.34. *There exists a real-valued nonnegative $\mathcal{C}^{2,\alpha}$-smooth function $\mu = \mu(e^{i\theta} : s)$ defined for $e^{i\theta} \in \partial\Delta$ and $s \in [-1,1]$ of support concentrated near $\{-1\} \times [-1,1]$ such that the mapping*

$$\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \widehat{X}'_{r_0,t:s}}{\partial \theta}\left(e^{i\theta}\right)\bigg|_{\theta=0} \in \mathbb{R}^{n-1} \tag{4.35}$$

is maximal equal to $(n-1)$ at $t=0$.

Geometrically speaking, since the vector $\frac{\partial \widehat{X}_{1;r_0:s}}{\partial\theta}\left(e^{i\theta}\right)\Big|_{\theta=0}$ is nonzero, it follows that when the parameter t varies, then the set of lines generated by the vectors $\frac{\partial \widehat{X}_{r_0,t:s}}{\partial\theta}\left(e^{i\theta}\right)\Big|_{\theta=0}$ covers an open cone in the space $T_{p_1}M^1 \equiv \mathbb{R}^n$ equipped with coordinates (x_1, x'), *see* again FIGURE 7 above for an illustration.

4.36. Adding pivoting and translation parameters. Let $\chi = (\chi_1, \chi') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\nu \in \mathbb{R}$ satisfying $|\chi| < \varepsilon$ and $|\nu| < \varepsilon$ for some small $\varepsilon > 0$. Then the mapping

$$\begin{cases} \mathbb{R}^{n+1} \ni (\chi_1, \chi', \nu) \longmapsto \big(\, \chi_1 + i[h_1(\chi : s) + \nu], \chi' + i\varphi'(\chi, h_1(\chi : s) + \nu : s)\,\big) \\ \qquad\qquad\qquad\qquad\qquad =: \widehat{p}(\chi, \nu : s) \in M_s \end{cases} \tag{4.37}$$

is a $\mathcal{C}^{2,\alpha}$-smooth diffeomorphism onto a neighborhood of the origin in M_s with the property that

(a) $\nu > 0$ if and only if $\widehat{p}(\chi,\nu:s) \in (M_s^1)^+$.
(b) $\nu = 0$ if and only if $\widehat{p}(\chi,\nu:s) \in M_s^1$.
(c) $\nu < 0$ if and only if $\widehat{p}(\chi,\nu:s) \in (M_s^1)^-$.

If $\tau \in \mathbb{R}$ with $|\tau| < \varepsilon$ is a supplementary parameter, we may now define a crucial deformation of the first component $\widehat{Z}_{1;r_0:s}\left(e^{i\theta}\right)$ by setting

$$\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}\left(e^{i\theta}\right) := ir_0\left(1-e^{i\theta}\right)[1+i\tau] + \chi_1 + i[h_1(\chi:s)+\nu]. \tag{4.38}$$

Of course, we have $\widehat{Z}_{1;r_0,0,0,0:s}\left(e^{i\theta}\right) \equiv \widehat{Z}_{1;r_0:s}\left(e^{i\theta}\right)$. Geometrically speaking, this perturbation corresponds to add firstly a small "rotation parameter" τ which rotates (and slightly dilates) the disc $ir_0\left(1-e^{i\theta}\right)$ passing through the origin in $\mathbb{C}_{z_1}$, to add secondly a small "translation parameter (χ_1, χ') which will enable to cover a neighborhood of the origin in M_s^1 and to add thirdly a small translation parameter ν along the y_1-axis.

Consequently, with this first component $\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}\left(e^{i\theta}\right)$, we can construct a $\mathbb{C}^n$-valued analytic disc $\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(\zeta)$ satisfying the important property

$$\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(1) = \widehat{p}(\chi, \nu : s), \tag{4.39}$$

simply by solving the perturbed Bishop type equation which extends (4.31)

$$\widehat{X}'_{r_0,t,\tau,\chi,\nu:s}\left(e^{i\theta}\right) = -\left[T_1\left(\Phi'\left(\widehat{Z}_{1;r_0,\tau,\chi,\nu:s}(\cdot), \widehat{X}'_{r_0,t,\tau,\chi,\nu:s}(\cdot), t\mu(\cdot : s) : s\right)\right)\right]\left(e^{i\theta}\right). \tag{4.40}$$

Of course, thanks to the sympathetic stability of Bishop's equation under perturbation, the solution exists and satisfies smoothness properties entirely similar to the ones stated after (4.32). We can summarize the description of our final family of analytic discs

$$\widehat{Z}_{r_0,t,\tau,\chi,\nu:s}(\zeta) : \begin{cases} r_0 = \text{approximate radius.} \\ t = \text{normal deformation parameter.} \\ \tau = \text{pivoting parameter.} \\ \chi = \text{parameter of translation along } M^1. \\ \nu = \text{parameter of translation in } M \text{ transversally to } M^1. \\ s = \text{parameter of the characteristic curve } \gamma. \\ \zeta = \text{unit disc variable.} \end{cases} \tag{4.41}$$

For every t and every χ, we now want to adjust the pivoting parameter τ in order that the disc boundary $\widehat{Z}_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)$ for $\nu = 0$ is tangent to M_s^1. This tangency condition will be useful in order to derive the crucial Lemma 4.51 below.

Lemma 4.42. *Shrinking ε if necessary, there exists a unique $\mathcal{C}^{1,\alpha-0}$-smooth map $(t, \chi, s) \mapsto \tau(t, \chi : s)$ defined for $|t| < \varepsilon$, for $|\chi| < \varepsilon$ and for $s \in [-1, 1]$ satisfying $\tau(0, 0 : s) = \partial_{t_j}\tau(0, 0 : s) = \partial_{\chi_k}\tau(0, 0 : s) = 0$ for $j = 1, \dots, n-1$ and $k = 1, \dots, n$, such that the vector*

$$\left.\frac{\partial}{\partial\theta}\right|_{\theta=0} \widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,0:s}\left(e^{i\theta}\right) \tag{4.43}$$

is tangent to M_s^1 at the point $\widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,0:s}(1) = \widehat{p}(\chi, 0 : s) \in M_s^1$.

Proof. We remind that M_s is represented by the $(n-1)$ scalar equations $y' = \varphi'(x, y_1 : s)$ and that M_s^1 is represented by the n equations $y_1 = h_1(x : s)$ and $y' = \varphi'(x, h_1(x : s) : s) \equiv h'(x' : s)$. We can therefore compute the Cartesian equations of the tangent plane to M_s^1 at the point $\widehat{p}(\chi, 0 : s) = \chi + ih(\chi : s))$:

$$\begin{cases} \mathsf{Y}_1 - h_1(\chi : s) = \sum_{k=1}^{n} \partial_{x_k} h_1(\chi : s)\, [\mathsf{X}_x - \chi_k]\,, \\ \mathsf{Y}' - \varphi'(\chi, h_1(\chi : s) : s) = \sum_{k=1}^{n} \left(\partial_{x_k}\varphi' + \partial_{y_1}\varphi' \cdot \partial_{x_k} h_1\right)[\mathsf{X}_k - \chi_k]\,. \end{cases} \tag{4.44}$$

On the other hand, we observe that the tangent vector

$$\left.\frac{\partial}{\partial\theta}\right|_{\theta=0} \widehat{Z}_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right) = \left(r_0[1 + i\tau],\ \left.\frac{\partial}{\partial\theta}\right|_{\theta=0} \widehat{Z}'_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)\right) \tag{4.45}$$

is already tangent to M_s at the point $\widehat{p}(\chi, 0 : s)$, because $M_{s,t} \equiv M_s$ in a neighborhood of the origin. More precisely, since $\Phi' \equiv \varphi'$ in a neighborhood of the origin, we may

differentiate with respect to θ at $\theta = 0$ the relation

$$\widehat{Y}'_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right) \equiv \varphi'\left(\widehat{X}_{r_0,\tau,\chi,0:s}\left(e^{i\theta}\right), \widehat{Y}_{1;r_0,\tau,\chi,0:s}\left(e^{i\theta}\right),\right) \tag{4.46}$$

which is valid for $e^{i\theta}$ close to 1 in $\partial\Delta$, noticing in advance that it follows immediately from (4.38) that

$$\frac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{X}_{1;r_0,\tau,\chi,0:s}\left(e^{i\theta}\right) = r_0 \quad \text{and} \quad \frac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{Y}_{1;r_0,\tau,\chi,0:s}\left(e^{i\theta}\right) = r_0\tau, \tag{4.47}$$

hence we obtain by a direct application of the chain rule

$$\begin{cases} \dfrac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{Y}'_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right) = \partial_{y_1}\varphi' \cdot r_0\tau + \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n} \partial_{x_k}\varphi' \cdot \left(\frac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{X}_{k;r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)\right). \end{cases} \tag{4.48}$$

On the other hand, the vector (4.45) belongs to the tangent plane to M_s^1 at $\widehat{p}(\chi, 0 : s)$ whose equations are computed in (4.44) if and only if the following two conditions are satisfied

$$\begin{cases} r_0\tau = \displaystyle\sum_{k=1}^{n} \partial_{x_k} h_1(\chi : s) \left[\frac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{X}_{k;r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)\right], \\ \dfrac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{Y}'_{r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right) = \\ \qquad = \displaystyle\sum_{k=1}^{n} \left(\partial_{x_k}\varphi' + \partial_{y_1}\varphi' \cdot \partial_{x_k} h_1\right) \cdot \left[\frac{\partial}{\partial\theta}\bigg|_{\theta=0} \widehat{X}_{k;r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)\right]. \end{cases} \tag{4.49}$$

We observe that the first line of (4.49) together with the relation (4.48) already obtained implies the second line of (4.49) by an obvious linear combination. Consequently, the vector (4.45) belongs to the tangent plane to M_s^1 at $\widehat{p}(\chi, 0 : s)$ if and only if the first line of (4.49) is satisfied. As r_0 is nonzero, as the first order derivative $\partial_{x_k} h_1(\chi : s)$ are of class $\mathcal{C}^{1,\alpha}$ and vanish at $x = 0$ and as $\frac{\partial}{\partial\theta}\big|_{\theta=0} \widehat{X}_{k;r_0,t,\tau,\chi,0:s}\left(e^{i\theta}\right)$ is of class $\mathcal{C}^{1,\alpha-0}$ with respect to all variables (t, τ, χ, s), it follows from the implicit function theorem that there exists a unique solution $\tau = \tau(t, \chi : s)$ of the first line of (4.49) which satisfies in addition the normalization conditions $\tau(0, 0 : s) = \partial_{t_j}\tau(0, 0 : s) = \partial_{\chi_k}\tau(0, 0 : s) = 0$ for $j = 1, \dots, n-1$ and $k = 1, \dots, n$. This completes the proof of Lemma 4.42. □

We now define the analytic disc

$$\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta) := \widehat{Z}_{r_0,t,\tau(t,\chi:s),\chi,\nu:s}(\zeta). \tag{4.50}$$

Lemma 4.51. *Shrinking ε if necessary, the following two properties are satisfied*:

- **(1)** $\widehat{\mathcal{Z}}_{t,\chi,0:s}(\partial\Delta\backslash\{1\}) \subset (M_s^1)^+$ *for all t, χ, ν and s with $|t| < \varepsilon$, with $|\chi| < \varepsilon$, with $|\nu| < \varepsilon$ and with $-1 \leq s \leq 1$.*
- **(2)** *If ν satisfies $0 < \nu < \varepsilon$, then $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\partial\Delta) \subset (M_s^1)^+$ for all t, χ and s with $|t| < \varepsilon$, with $|\chi| < \varepsilon$ and with $-1 \leq s \leq 1$.*

Proof. To establish property **(1)**, we first observe that the disc $\widehat{\mathcal{Z}}_{0,0,0:s}\left(e^{i\theta}\right)$ identifies with the disc $\widehat{Z}_{r_0:s}(e^{i\theta})$ defined in §4.29. According to Lemma 4.25, we know that $\widehat{\mathcal{Z}}_{0,0,0:s}(\partial\Delta\backslash\{1\})$ is contained in $(M_s^1)^+$. By continuity, if ε is sufficiently small, we can assume that for all t with $|t| < \varepsilon$, for all χ with $|\chi| < \varepsilon$ and for all θ with $\frac{\pi}{4} \leq |\theta| \leq \pi$,

the point $\widehat{\mathcal{Z}}_{t,\chi,0:s}\left(e^{i\theta}\right)$ is contained in $(M_s^1)^+$. It remains to control the part of $\partial\Delta$ which corresponds to $|\theta| \leq \frac{\pi}{4}$.

Since the disc $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}\left(e^{i\theta}\right)$ is of class $\mathcal{C}^2$ with respect to all its principal variables $\left(t,\chi,\nu,e^{i\theta}\right)$, if $|t| < \varepsilon$, if $|\chi| < \varepsilon$ and if $0 \leq |\theta| \leq \frac{\pi}{4}$, for sufficiently small ε, then the inequalities (4.28) are just perturbed a little bit, so we can assume that

$$\text{(4.52)} \qquad \begin{cases} \partial_\theta\partial_\theta\widehat{\mathcal{Y}}_{1;t,\chi,0:s}\left(e^{i\theta}\right) \geq r_0, \\ \left|\partial_\theta\partial_\theta h_1\left(\widehat{\mathcal{X}}_{t,\chi,0:s}\left(e^{i\theta}\right)\right)\right| \leq \dfrac{r_0}{2}. \end{cases}$$

We claim that the inequality

$$\text{(4.53)} \qquad \widehat{\mathcal{Y}}_{1;t,\chi,0:s}\left(e^{i\theta}\right) > \left|h_1\left(\widehat{\mathcal{X}}_{t,\chi,0:s}\left(e^{i\theta}\right)\right)\right|$$

holds for all $0 < |\theta| \leq \frac{\pi}{4}$, which will complete the proof of property **(1)**.

Indeed, we first remind that the tangency to M_s^1 of the vector $\frac{\partial}{\partial\theta}\big|_{\theta=0}\widehat{\mathcal{Z}}_{t,\chi,0:s}\left(e^{i\theta}\right)$ at the point $\widehat{p}(\chi, 0 : s)$ is equivalent to the first relation (4.49), which may be rewritten in terms of the components of the disc $\widehat{\mathcal{Z}}_{t,\chi,0:s}\left(e^{i\theta}\right)$ as follows

$$\text{(4.54)} \qquad \partial_\theta\widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) = \sum_{k=1}^{n}\left[\partial_{x_k} h_1\right]\left(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\right)\cdot\partial_\theta\widehat{\mathcal{X}}_{k;t,\chi,0:s}(1).$$

Substracting this relation from (4.53) and substracting also the relation $\widehat{Y}_{1;t,\chi,0:s}(1) = h_1\left(\mathcal{X}_{t,\chi,0:s}(1)\right)$, we see that it suffices to establish that for all θ with $0 < |\theta| \leq \frac{\pi}{4}$, we have the strict inequality

$$\text{(4.55)} \qquad \begin{cases} \widehat{\mathcal{Y}}_{1;t,\chi,0:s}\left(e^{i\theta}\right) - \widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) - \theta\cdot\partial_\theta\widehat{\mathcal{Y}}_{1;t,\chi,0:s}(1) > \\ \quad > \Bigg|h_1\left(\widehat{\mathcal{X}}_{t,\chi,0:s}\left(e^{i\theta}\right)\right) - h_1\left(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\right) - \\ \qquad\qquad - \displaystyle\sum_{k=1}^{n}\left[\partial_{x_k} h_1\right]\left(\widehat{\mathcal{X}}_{t,\chi,0:s}(1)\right)\cdot\partial_\theta\widehat{\mathcal{X}}_{k;t,\chi,0:s}(1)\Bigg| \end{cases}$$

However, by means of Taylor's integral formula, this last inequality may be rewritten as

$$\text{(4.56)} \qquad \int_0^\theta(\theta-\theta')\cdot\partial_\theta\partial_\theta\widehat{\mathcal{Y}}_{1;t,\chi,0:s}\left(e^{i\theta'}\right)\cdot d\theta' > \left|\int_0^\theta(\theta-\theta')\cdot\partial_\theta\partial_\theta\left[h_1\left(\widehat{\mathcal{X}}_{t,\chi,0:s}\left(e^{i\theta'}\right)\right)\right]\cdot d\theta'\right|$$

and it follows immediately by means of (4.52).

Secondly, to check property **(2)**, we observe that by the definition (4.37), the parameter ν corresponds to a translation of the z_1-component of the disc boundary $\widehat{\mathcal{Z}}_{t,\chi,0:s}(\partial\Delta)$ along the y_1 axis. More precisely, we have

$$\text{(4.57)} \qquad \frac{\partial}{\partial\nu}\widehat{\mathcal{Y}}_{1;t,\chi,\nu:s}(\zeta) \equiv 1, \qquad\qquad \frac{\partial}{\partial\nu}\widehat{\mathcal{X}}_{1;t,\chi,\nu:s}(\zeta) \equiv 0.$$

On the other hand, differentiating Bishop's equation (4.40), and using the smallness of the function Φ', it may be checked that

$$\text{(4.58)} \qquad \left|\frac{\partial}{\partial\nu}\widehat{Z}'_{r_0,t,\tau,\chi,\nu:s}(e^{i\theta})\right| << 1,$$

if r_0 and ε are sufficiently small. It follows that the disc boundary $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\partial\Delta)$ is globally moved in the direction of the y_1-axis as $\nu > 0$ increases, hence is contained in $(M_s^1)^+$.

The proof of Lemma 4.51 is complete. □

4.59. Local half-wedges. As a consequence of Lemma 4.34, of (4.39) and of property **(2)** of Lemma 4.51, we conclude that for every $s \in [-1, 1]$, our discs $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)$ satisfy all the requirements **(i)**, **(ii)** and **(iii)** of §4.2 insuring that the set defined by

$$\mathcal{HW}_s^+ := \left\{ \widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\rho) : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ 0 < \nu < \varepsilon, \ 1 - \varepsilon < \rho < 1 \right\} \tag{4.60}$$

is a local half-wedge of edge $(M_s^1)^+$ at the origin in the $\widehat{z}$-coordinates, which corresponds to the point $\gamma(s)$ in the z-coordinates. Coming back to the z coordinates, we may define the family of analytic discs

$$\mathcal{Z}_{t,\chi,\nu:s}(\zeta) := \Phi_s^{-1} \left(\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta) \right) \tag{4.61}$$

and we shall construct the desired semi-local attached half-wedge $\mathcal{HW}_\gamma^+$ of Proposition 4.6.

4.62. Holomorphic extension to an attached half-wedge. Indeed, we can now complete the proof of Proposition 4.6. Let us denote by $\widehat{z} = \Phi_s(z)$ the parametrized change of coordinates defined in §4.11, where the point $\gamma(s)$ in z-coordinates corresponds to the origin in $\widehat{z}$-coordinates. Given an arbitrary holomorphic function $f \in \mathcal{O}(\Omega)$ as in Proposition 4.6, by the change of coordinates $\widehat{z} = \Phi_s(z)$ and by restriction to $(M_s^1)^+$, we get a CR function $\widehat{f}_s \in \mathcal{C}^0_{CR}\left((M_s^1)^+ \cap U_1\right)$, for some small neighborhood U_1 of the origin in $\mathbb{C}^n$, whose size is uniform with respect to s. Thanks to an obvious generalization of the approximation Lemma 4.8 with a supplementary parameter $s \in [-1, 1]$, we know that there exists a second uniform neighborhood $V_1 \subset\subset U_1$ of the origin in $\mathbb{C}^n$ such that every continuous CR function in $\mathcal{C}^0_{CR}\left((M_s^1)^+ \cap U_1\right)$ is uniformly approximable by polynomials on $(M_s^1)^+ \cap V_1$. In particular, this property holds for the CR function $\widehat{f}_s$. Furthermore, choosing r_0 and ε sufficiently small, we can insure that all the discs $\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)$ are attached to $(M_s^1)^+ \cap V_1$. It then follows from the maximum principle applied to the approximating sequence of polynomials that for each $s \in [-1, 1]$, the function $\widehat{f}_s$ extends holomorphically to the half-wedge defined by (4.60). Finally, we deduce that the holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically to the half-wedge attached to the one-sided neighborhood $(M_\gamma^1)^+$ defined by

$$\begin{cases} \mathcal{HW}_\gamma^+ := \{ \mathcal{Z}_{t,\chi,\nu:s}(\rho) : \ |t| < \varepsilon, \ |\chi| < \varepsilon, \ 0 < \nu < \varepsilon, \\ \qquad\qquad\qquad 1 - \varepsilon < \rho < 1, \ -1 \le s \le s \}. \end{cases} \tag{4.63}$$

Without shrinking Ω near the points $\mathcal{Z}_{t,\chi,\nu:s}(-1)$ (otherwise, the crucial rank property of Lemma 4.34 would degenerate), we can shrink the open set Ω in a very thin neighborhood of the characteristic segment γ in M and we can shrink $\varepsilon > 0$ if necessary in order that the intersection $\Omega \cap \mathcal{HW}_\gamma^+$ is connected. By the principle of analytic continuation, this implies that there exists a well-defined holomorphic function $F \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_\gamma^+\right)$ with $F|_\Omega = f$.

The proof of Proposition 4.6 is complete. □

4.64. Local half-wedge in CR dimension $m \geq 2$. Repeating the above constructions in the simpler case where the curve γ degenerates to the point p_1 of Theorem 3.22 and adding some supplementary parameters along the $(m-1)$ remaining complex tangent directions of M at p_1 for the constructions of analytic discs, we can show that under the assumptions of Theorem 3.22, there exists a local half-wedge $\mathcal{HW}_{p_1}^+$ at p_1 to which (shrinking Ω if necessary) every holomorphic function $f \in \mathcal{O}(\Omega)$ extends holomorphically. We shall not write down the details.

4.65. Transition. In the case $m = 2$, using such a local half-wedge and applying the continuity principle along analytic discs of whose one boundary part is contained in M and whose second boundary part is contained in the local half-wedge $\mathcal{HW}^+_{p_1}$, we shall establish that p_1 is $\mathcal{W}$-removable in Section 10 below. At present, in the more delicate case $m = 1$, we shall pursue in Section 5 below our geometric constructions for the choice of a special point $p_{\rm sp} \in C$ which satisfies the conclusion of Theorem 3.19 **(i)**.

§5. CHOICE OF A SPECIAL POINT OF $C_{\rm nr}$ TO BE REMOVED LOCALLY

5.1. Choice of a first supporting hypersurface. Continuing with the proof of Theorem 3.19 **(i)**, we shall now analyze and use the important geometric condition $\mathcal{F}^c_{M^1}\{C\}$ defined in Theorem 1.2'. We first delineate a convenient geometric situation.

Lemma 5.2. *Under the assumptions of Theorem 3.19* **(i)**, *there exists a $\mathcal{C}^{2,\alpha}$-smooth segment $\gamma : [-1, 1] \to M^1$ running in characteristic directions, namely satisfying $d\gamma(s)/ds \in T^c_{\gamma(s)}M \cap T_{\gamma(s)}M^1 \backslash \{0\}$ such that $\gamma(-1) \not\in C$, $\gamma(0) \in C$, $\gamma(1) \not\in C$, and there exists a $\mathcal{C}^{1,\alpha}$-smooth hypersurface H^1 of M^1 with $\gamma \subset H^1$ which is foliated by characteristic segments close to γ, such that locally in a neighborhood of H^1, the closed subset C is contained in $\gamma \cup (H^1)^-$, where $(H^1)^-$ denotes an open one-sided neighborhood of H^1 in M^1.*

Proof. By the assumption $\mathcal{F}^c_{M^1}\{C\}$, there exists a characteristic curve $\widetilde{\gamma} : [-1, 1] \to M^1$ with $\widetilde{\gamma}(-1) \not\in C$, $\widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \not\in C$, there exists a neighborhood $V^1_{\widetilde{\gamma}}$ of $\widetilde{\gamma}$ in M^1 and there exists a local $(n-1)$-dimensional submanifold R^1 passing through $\widetilde{\gamma}(0)$ which is transversal to $\widetilde{\gamma}$ such that the semi-local projection $\pi_{\mathcal{F}^c_{M^1}} : V^1_{\widetilde{\gamma}} \to R^1$ parallel to the characteristic curves maps C onto the closed subset $\pi_{\mathcal{F}^c_{M^1}}(C)$ with the property that $\pi_{\mathcal{F}^c_{M^1}}(\widetilde{\gamma})$ lies on the *boundary of $\pi_{\mathcal{F}^c_{M^1}}(C)$ with respect to the topology of R^1*. This property is illustrated in the right hand side of the following figure.

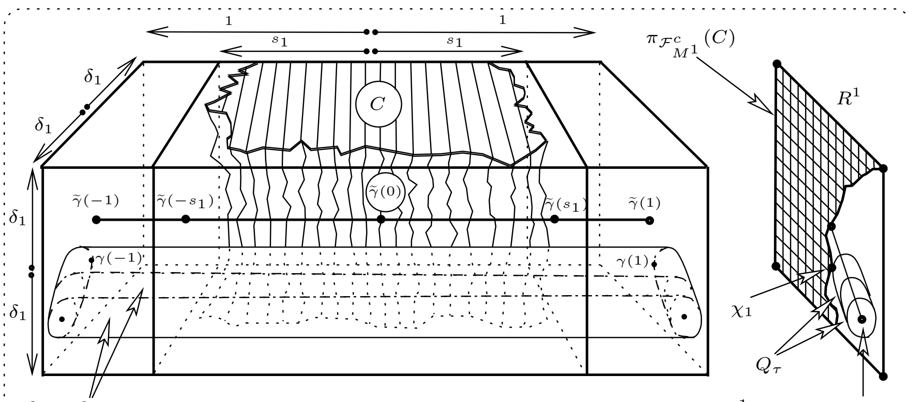

FIGURE 8: CONSTRUCTION OF A SUPPORT HYPERSURFACE H^1

However, we want in addition a foliated supporting hypersurface H^1, which does not necessarily exist in a neighborhood of $\widetilde{\gamma}$. To construct H^1, let us first straighten the characteristic lines in a neighborhood of $\widetilde{\gamma}$, getting a product $[-1, 1] \times [-\delta_1, \delta_1]^{n-1}$, for some $\delta_1 > 0$, equipped with coordinates $(s, \chi) = (s, \chi_2, \dots, \chi_n) \in \mathbb{R} \times \mathbb{R}^{n-1}$, so that the lines $\{\chi = {\rm ct.}\}$ correspond to characteristic lines. Such a straightening is only of class $\mathcal{C}^{1,\alpha}$, because the line distribution $T^cM|_{M^1} \cap TM^1$ is only of class $\mathcal{C}^{1,\alpha}$. Clearly,

we may assume that δ_1 is so small that there exists s_1 with $0 < s_1 < 1$ such that the two cubes $[-1, -s_1] \times [-\delta_1, \delta_1]^{n-1}$ and $[s_1, 1] \times [-\delta_1, \delta_1]^{n-1}$ do not meet the singularity C, as drawn in FIGURE 8 above.

We may identify the transversal R^1 with $[-\delta_1, \delta_1]^{n-1}$; then the projection of $\widetilde{\gamma}$ is the origin of R^1. By assumption, $\pi_{\mathcal{F}^c_{M^1}}(C)$ is a proper closed subset of R^1 with the origin lying on its boundary. We can therefore choose a point χ_0 in the interior of R^1 lying outside $\pi_{\mathcal{F}^c_{M^1}}(C)$. Also, we can choose a small open $(n-1)$-dimensional ball Q_0 centered at this point which is contained in the complement $R^1 \backslash \pi_{\mathcal{F}^c_{M^1}}(C)$. Furtermore, we can include this ball in a one parameter family of $\mathcal{C}^{1,\alpha}$-smooth domains $Q_\tau \subset R^1$, for $\tau \geq 0$, which are parts of ellipsoids stretched along the segment which joins the point χ_0 with the origin of R^1.

We then consider the tube domains $[-1, 1] \times Q_\tau$ in $[-1, 1] \times [-\delta_1, \delta_1]^{n-1}$. Clearly, there exists the smallest $\tau_1 > 0$ such that the tube $[-1, 1] \times Q_{\tau_1}$ meets the singularity C on its boundary $[-1, 1] \times \partial Q_{\tau_1}$. In particular, there exists a point $\chi_1 \in \partial Q_{\tau_1}$ such that the characteristic segment $[-1, 1] \times \{\chi_1\}$ intersects C. Increasing a little bit the curvature of ∂Q_{τ_1} in a neighborhood of χ_1 if necessary, we can assume that $\pi_{\mathcal{F}^c_{M^1}}(C) \cap \overline{Q_{\tau_1}} = \{\chi_1\}$ in a neighborhood of χ_1. Moreover, since by construction the two segments $[-1, -s_1] \times \{\chi_1\} \cup [s_1, 1] \times \{\chi_1\}$ do not meet C, we can reparametrize the characteristic segment $[-1, 1] \times \{\chi_1\}$ as $\gamma : [-1, 1] \to M^1$ with $\gamma(-1) \notin C$, $\gamma(0) \in C$ and $\gamma(1) \notin C$. Since all characteristic lines are $\mathcal{C}^{2,\alpha}$-smooth, we can choose the parametrization to be of class $\mathcal{C}^{2,\alpha}$. For the supporting hypersurface H^1, it suffices to choose a piece of $[-1, 1] \times \partial Q_{\tau_1}$ near $[-1, 1] \times \{\chi_1\}$. By construction, this supporting hypersurface is only of class $\mathcal{C}^{1,\alpha}$ and we have that C is contained in $\gamma \cup (H^1)^-$ semi-locally in a neighborhood of γ, as desired. This completes the proof of Lemma 5.2. □

5.3. Field of cones on M^1. With the characteristic segment γ constructed in Lemma 5.2, by an application of Proposition 4.6, we deduce that there exists a semi-local half-wedge $\mathcal{HW}^+_\gamma$ attached to $(M^1_\gamma)^+ \cap V_\gamma$, for some neighborhood V_γ of γ in M, to which $\mathcal{O}(\Omega)$ extends holomorphically.

Then, we remind that by (4.37), (4.39) and (4.50), for all t with $|t| < \varepsilon$, the point $\widehat{\mathcal{Z}}_{t,\chi,0:s}(1)$ identifies with the point $\widehat{p}(\chi, 0 : s) \in M^1_s$ defined in (4.37) (which is independent of t) and the mapping $\chi \mapsto \widehat{\mathcal{Z}}_{t,\chi,0:s}(1) \in M^1_s$ is a local diffeomorphism.

Sometimes in the sequel, we shall denote the disc $\mathcal{Z}_{t,\chi,\nu:s}(\zeta) \equiv \Phi_s^{-1}\left(\widehat{\mathcal{Z}}_{t,\chi,\nu:s}(\zeta)\right)$ defined in (4.61) by $\mathcal{Z}_{t,\chi_1,\chi',\nu:s}(\zeta)$, where $\chi' = (\chi_2, \dots, \chi_n) \in \mathbb{R}^{n-1}$. Since the characteristic curve is directed along the x_1-axis, which is transversal in $T_0 M^1_s$ to the space $\{(0, \chi')\}$, it follows that the mapping $(s, \chi') \longmapsto \mathcal{Z}_{t,0,\chi',0:s}(1) = \Phi_s^{-1}(\widehat{p}(0, \chi', 0 : s))$ is, independently of t, a diffeomorphism onto its image for $s \in [-1, 1]$ and for χ' close to the origin in $\mathbb{R}^{n-1}$. To fix ideas, we shall let χ' vary in the *closed* cube $[-\varepsilon, \varepsilon]^{n-1}$ (analogously to the fact that s runs in the *closed* interval $[-1, 1]$) and we shall denote by V^1_γ the closed subset of M^1 which is the image of this diffeomorphism.

At every point $p := \mathcal{Z}_{t,0,\chi',0:s}(1) = \mathcal{Z}_{0,0,\chi',0:s}(1)$ of this neighborhood V^1_γ, we define an open infinite oriented cone contained in the n-dimensional linear space $T_p M^1$ by

$$\mathsf{C}_p := \mathbb{R}^+ \cdot \left\{ \frac{\partial \mathcal{Z}_{t,0,\chi',0:s}}{\partial \theta}(1) : \ |t| < \varepsilon \right\}. \tag{5.4}$$

The fact that C_p is indeed an open cone follows from Lemma 4.34, from (4.61) and from the fact that Φ_s^{-1} is a biholomorphism. This cone contains in its interior the nonzero

vector

$$v_p^0 := \frac{\partial \mathcal{Z}_{0,0,\chi',0:s}}{\partial \theta}(1) \in \mathsf{C}_p \subset T_pM^1 \backslash \{0\}. \tag{5.5}$$

We shall say that the cone C_p is the *cone created at* p *by the semi-local attached half-wedge* $\mathcal{HW}_\gamma^+$ (more precisely, by the family of analytic discs which covers this semi-local half-wedge).

As p varies, the mapping $p \mapsto \mathsf{C}_p$ constitutes what we shall call a *field of cones over* V_γ^1, *see* FIGURE 2 in Section 2 above and FIGURE 9 just below for illustrations.

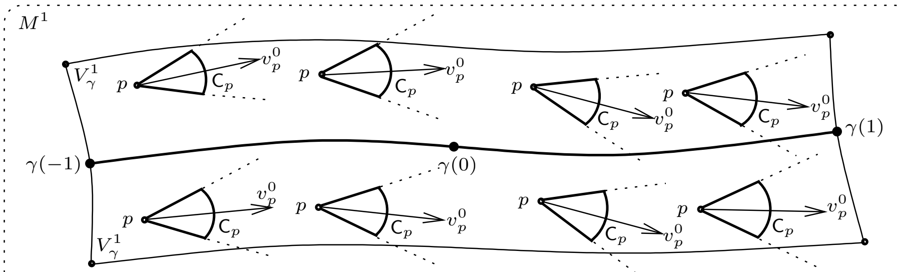

FIGURE 9: FIELD OF CONES ON TM^1 ASSOCIATED TO THE FAMILY $\mathcal{Z}_{t,0,\chi',0,s}(\zeta)$

The mapping $p \mapsto v_p^0$ defines a $\mathcal{C}^{1,\alpha-0}$-smooth vector field tangent to M^1. This vector field is contained in the field of cones $p \mapsto \mathsf{C}_p$. Over V_γ^1, we can also consider a nowhere zero characteristic vector field X whose direction agrees with the orientation of γ and which satisfyies $\exp(sX)(\gamma(0)) = \gamma(s)$ for all $s \in [-1,1]$. Furthermore, for every $p \in V_\gamma^1$, we define the *filled cone*

$$\mathsf{FC}_p := \mathbb{R}^+ \cdot \{\lambda \cdot X_p + (1-\lambda) \cdot v_p : \ 0 \leq \lambda < 1, \ v_p \in \mathsf{C}_p\}. \tag{5.6}$$

In the right hand side of FIGURE 10 just below, in the tangent space T_pM^1 equipped with linear coordinates $(x_1, \dots, x_n)$ such that the characteristic direction $T_p^cM \cap T_pM^1$ is directed along the x_1-axis, we draw C_p, its filling FC_p and its projection $\pi'(\mathsf{C}_p)$ onto the $(x_2, , \dots, x_n)$-space parallel to the x_1-axis.

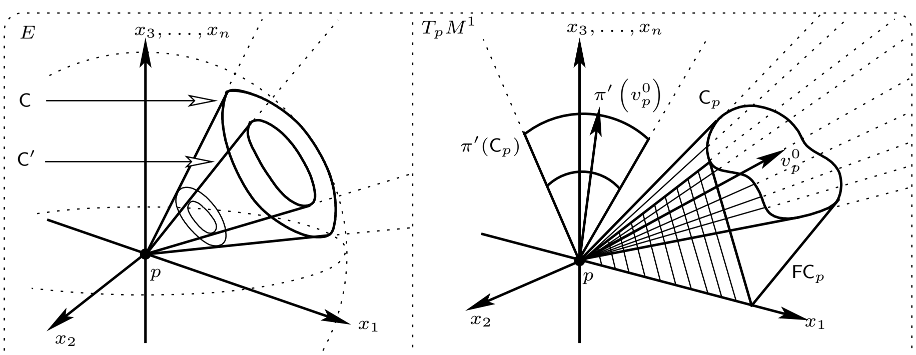

FIGURE 10: A CONE IN T_pM^1 AND ITS FILLING ALONG THE CHARACTERISTIC DIRECTION

Given an arbitrary nonzero vector $v_p \in \mathsf{C}_p$, where $p \in V_\gamma^1$, it may be checked that a small neighborhood of the origin in the positive half-line $\mathbb{R}^+ \cdot Jv_p$ generated by Jv_p, where J denotes the complex structure of $T\mathbb{C}^n$, is contained in the attached half-wedge

$\mathcal{HW}_\gamma^+$. More generally, the same property holds for every nonzero vector v_p which belongs to the filled cone FC_p. In fact, we shall establish that analytic discs which are half-attached to M^1 at p, namely send $\partial^+\Delta := \{\zeta \in \partial\Delta : \operatorname{Re}\zeta \geq 0\}$ to M^1, which are closed to the complex line $v_p + Jv_p$ in $\mathcal{C}^1$ norm and which are sufficiently small are contained in the attached half-wedge $\mathcal{HW}_\gamma^+$. The interest of this property and the reason why we have defined fields of cones and their filling will be more appearant in Sections 8 and 9 below, where we apply the continuity principle to achieve the proof of Theorem 3.19 **(i)**. A precise statement, involving families of analytic discs $A_c(\zeta)$ which will be constructed in Section 7 below, is as follows. For the proof of a more precise statement involving families of analytic discs $A^1_{x,v:c}(\zeta)$, we refer to Section 7 and especially to Lemma 8.3 **($\mathbf{9_1}$)** below.

Lemma 5.7. *Fix a point $p \in V_\gamma^1$ and a vector v_p in the cone C_p created by the semi-local attached half-wedge $\mathcal{HW}_\gamma^+$ at p. Suppose that there exist two constants $c_1 > 0$ and $\Lambda_1 > 1$ such that for every c with $0 < c \leq c_1$, there exists a $\mathcal{C}^{2,\alpha-0}$-smooth analytic disc $A_c(\zeta)$ with $A_c(\partial^+\Delta) \subset M^1$, such that*

- **(i)** *The positive half-line generated by the boundary of A_c at $\zeta = 1$ coincides with the positive half-line generated by v_p, namely $\mathbb{R}^+ \cdot \frac{\partial A_c}{\partial\theta}(1) \equiv \mathbb{R}^+ \cdot v_p$.*
- **(ii)** $|A_c(\zeta)| \leq c^2 \cdot \Lambda_1$ *for all* $\zeta \in \overline{\Delta}$ *and* $c \cdot \frac{1}{\Lambda_1} \leq \left|\frac{\partial A_c}{\partial\theta}(1)\right| \leq c \cdot \Lambda_1$.
- **(iii)** $\left|\frac{\partial A_c}{\partial\theta}\left(\rho e^{i\theta}\right) - \frac{\partial A_c}{\partial\theta}(1)\right| \leq c^2 \cdot \Lambda_1$ *for all* $\zeta = \rho e^{i\theta} \in \overline{\Delta}$.

If c_1 is sufficiently small, then for every c with $0 < c \leq c_1$, the set $A_c\left(\overline{\Delta}\backslash\partial^+\Delta\right)$ is contained in the semi-local half-wedge $\mathcal{HW}_\gamma^+$. Furthermore, the same conclusion holds if the nonzero vector v_p belongs to the filled cone FC_p.

5.8. Choice of the special point $p_{\rm sp}$ in the CR dimension $m = 1$ case. We can now answer the question raised after the statement of Theorem 3.19 **(i)**, which was the main purpose of the present Section 5: *How to choose the special point $p_{\rm sp}$ to be removed locally*? In the following statement, property **(ii)** will be really crucial for the removal of $p_{\rm sp}$, *see* in advance Proposition 5.12 below.

Lemma 5.9. *Let γ be the characteristic segment constructed in Lemma 5.2 above. Let $\mathcal{HW}_\gamma^+$ be the semi-local attached half-wedge of edge $(M_\gamma^1)^+ \cap V_\gamma$ constructed in Proposition 4.6 above, and let $p \mapsto \mathsf{FC}_p$ be the filled field of cones created in a closed neighborhood V_γ^1 of γ in M^1 by this semi-local attached half-wedge $\mathcal{HW}_\gamma^+$. Then there exists a special point $p_{\rm sp} \in V_\gamma^1$ such that the following two geometric properties are fulfilled*:

- **(i)** *There exists a $\mathcal{C}^{2,\alpha}$-smooth local supporting hypersurface $H_{\rm sp}$ of M^1 passing through $p_{\rm sp}$ such that, locally in a neighborhood of $p_{\rm sp}$, the closed subset C is contained in $(H_{\rm sp})^- \cup \{p_{\rm sp}\}$, where $(H_{\rm sp})^-$ denotes an open one-sided neighborhood of $H_{\rm sp}$ in M^1.*
- **(ii)** *There exists a nonzero vector $v_{\rm sp} \in T_{p_{\rm sp}}H_{\rm sp}$ which belongs to the filled cone $\mathsf{FC}_{p_{\rm sp}}$.*

Proof. According to Lemma 5.2, the singularity C is contained in $\gamma \cup (H^1)^-$, where H^1 is a $\mathcal{C}^{1,\alpha}$-smooth hypersurface containing γ which is foliated by characteristic segments. If $\lambda \in [0,1)$ is very close to 1, the vector field over V_γ^1 defined by

$$p \longmapsto v_p^\lambda := \lambda \cdot X_p + (1-\lambda) \cdot v_p \in T_pM^1 \tag{5.10}$$

is very close to the characteristic vector field X_p, so the integral curves of $p \mapsto v_p^\lambda$ are very close to the integral curves of $p \mapsto X_p$, which are the characteristic segments. If

λ is sufficiently close to 1, we can choose a subneighborhood $V_\gamma^\lambda \subset V_\gamma^1$ of γ which is foliated by integral curves of $p \mapsto v_p^\lambda$. As in Lemma 5.2, let us fix an $(n-1)$-dimensional submanifold R^1 transversal to γ and passing through $\gamma(0)$. Since the vector field $p \mapsto v_p^\lambda$ is very close to the characteristic vector field, it follows that after projection onto R^1 parallelly to the integral curves of $p \mapsto v_p^\lambda$, the closed set $C \cap V_\gamma^\lambda$ is again a proper closed subset of R^1. We notice that, by its very definition, the vector v_p^λ belongs to the filled cone FC_p for all $p \in V_\gamma^\lambda$.

We can proceed exactly as in the proof of Lemma 5.2 with the foliation of V_γ^λ induced by the integral curves of the vector field $p \mapsto v_p^\lambda$, instead of the characteristic foliation, except that we want a supporting hypersurface $H_{\rm sp}$ which is of class $\mathcal{C}^{2,\alpha}$. Consequently, we first approximate the vector field $p \mapsto v_p^\lambda$ by a new vector field $p \mapsto \widetilde{v}_p^\lambda$ whose coefficients are of class $\mathcal{C}^{2,\alpha}$ (with respect to every local graphing function of M^1) and which is very close to the vector field $p \mapsto v_p^\lambda$ in $\mathcal{C}^1$-norm. Again, we get a subneighborhood $\widetilde{V}_\gamma^\lambda \subset V_\gamma^\lambda$ of γ which is foliated by integral curves of $p \mapsto \widetilde{v}_p^\lambda$ and a projection of $C \cap \widetilde{V}_\gamma^\lambda$ which is a proper closed subset of R^1. Moreover, if the approximation is sufficiently sharp, we still have $\widetilde{v}_p^\lambda \in \mathsf{FC}_p$ for all $p \in \widetilde{V}_\gamma^\lambda$. Then by repeating the reasoning which yielded Lemma 5.2, using the foliation of $\widetilde{V}_\gamma^\lambda$ induced by $p \mapsto \widetilde{v}_p^\lambda$, we deduce that there exists an integral curve $\widetilde{\gamma}$ of the vector field $p \mapsto \widetilde{v}_p^\lambda$ satisfying (after reparametrization) $\widetilde{\gamma}(-1) \notin C$, $\widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \notin C$, together with a $\mathcal{C}^{2,\alpha}$-smooth supporting hypersurface $\widetilde{H}$ of $\widetilde{V}_\gamma^\lambda$ which contains $\widetilde{\gamma}$ such that C is contained in $\widetilde{\gamma} \cup (\widetilde{H})^-$. The fact that $\widetilde{H}$ is of class $\mathcal{C}^{2,\alpha}$ is due to the $\mathcal{C}^{2,\alpha}$-smoothness of the foliation on $\widetilde{V}_\gamma^\lambda$ induced by the vector field $p \mapsto \widetilde{v}_\gamma^\lambda$. In FIGURE 11 just below, suited for the case where M^1 is two-dimensional, we have drawn as a dotted line the limiting integral curve $\widetilde{\gamma}$ of $p \mapsto \widetilde{v}_p^\lambda$ having the property that C lies in one closed side of $\widetilde{\gamma}$ in $\widetilde{V}_\gamma^\lambda$.

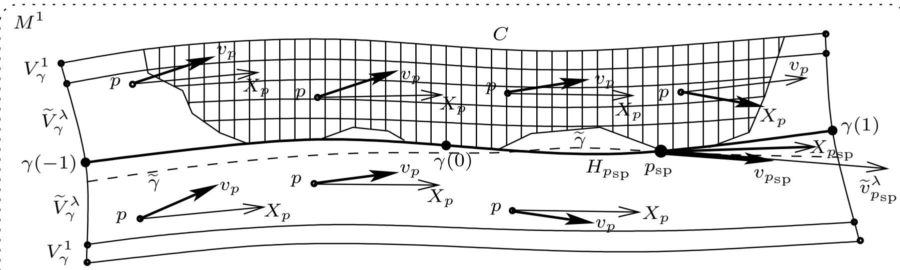

FIGURE 11: DOTTED INTEGRAL CURVE OF THE VECTOR FIELD $p \mapsto v_p^\lambda$ AND CHOICE OF $p_{\rm sp}$

To conclude the proof of Lemma 5.9, for the desired special point $p_{\rm sp}$, it suffices to choose $\widetilde{\gamma}(0)$. For the desired local supporting hypersurface $H_{p_{\rm sp}}$, we cannot choose directly a piece of $\widetilde{H}$ passing through $p_{\rm sp}$, because an open interval contained in $C \cap \widetilde{\gamma}$ may well be contained in $\widetilde{H}$ by the construction in the proof of Lemma 5.2 that we have just reapplied. Fortunately, since we know that locally in a neighborhood of $p_{\rm sp}$, the closed subset C is contained in $(\widetilde{H})^- \cup \widetilde{\gamma}$, it suffices to choose for the desired supporting hypersurface $H_{p_{\rm sp}} \subset M^1$ a piece of a $\mathcal{C}^{2,\alpha}$-smooth hypersurface passing through p_1, tangent to $\widetilde{H}$ at p_1 and satisfying $H_{p_{\rm sp}} \backslash \{p_{\rm sp}\} \subset (\widetilde{H})^+$ in a neighborhood of $p_{\rm sp}$. Finally, for

the nonzero vector $v_{\rm sp}$, it suffices to choose any positive multiple of the vector $\widetilde{v}_{p_{\rm sp}}^{\lambda}$. This completes the proof of Lemma 5.9. □

In Section 8 below, property **(ii)** of Lemma 5.9 together with the observation made in Lemma 5.7 will be crucial for the local $\mathcal{W}$-removability of the special point $p_{\rm sp}$.

5.11. Main removability proposition in the CR dimension $m = 1$ case. We can now formulate the main removability proposition to which Theorem 3.19 **(i)** is now reduced. From now on, we localize the situation at $p_{\rm sp}$, we denote this point simply by p_1, we denote its supporting hypersurface simply by H^1 and we denote its associated vector simply by v_1. Furthermore, we localize at p_1 the family of analytic discs considered in Section 4 for the construction of the semi-local attached half-wedge $\mathcal{HW}_\gamma^+$, hence we get a family of analytic discs $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ which satisfy properties **(i)**, **(ii)** and **(iii)** of §4.2 and which generate a local half-wedge $\mathcal{HW}_{p_1}^+ \subset \mathcal{HW}_\gamma^+$ as defined in (4.5). At present, we deduce from our constructions achieved in Section 4 and in the beginning of Section 5 that Theorem 3.19 **(i)** is now a consequence of the following main Proposition 5.12 just below, to the proof of which Sections 6, 7, 8 and 9 below are devoted. From now on, all our geometric considerations will be localized at the special point p_1.

Proposition 5.12. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of CR dimension $m = 1$ and of codimension $d = n-1 \geq 1$, let $M^1 \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold which is generic in $\mathbb{C}^n$, let $p_1 \in M^1$, let $H^1 \subset M^1$ be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M^1 passing through p_1 and let $(H^1)^-$ denote an open local one-sided neighborhood of H^1 in M^1. Let $C \subset M^1$ be a* nonempty *proper closed subset of M^1 with $p_1 \in C$ which is situated, locally in a neighborhood of p_1, only in one side of H^1, namely $C \subset (H^1)^- \cup \{p_1\}$. Let Ω be a neighborhood of $M \backslash C$ in $\mathbb{C}^n$, let $\mathcal{HW}_{p_1}^+$ be a local half-wedge of edge $(M^1)^+$ at p_1 generated by a family of analytic discs $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ satisfying the properties* **(i)**, **(ii)** *and* **(iii)** *of §4.2, let $\mathsf{C}_{p_1} \subset T_{p_1}M^1$ be the cone created by $\mathcal{HW}_{p_1}^+$ at p_1 and let FC_{p_1} be its filling.* As a main assumption, *suppose that there exists a nonzero vector $v_1 \in T_{p_1}H^1$ which belongs to the filled cone FC_{p_1}.*

- **(I)** *If v_1 does not belong to $T_{p_1}^c M$, then there exists a local wedge $\mathcal{W}_{p_1}$ of edge M at (p_1, Jv_1) with $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}_{p_1}^+\right]$ connected* (*shrinking $\Omega \cup \mathcal{HW}_{p_1}^+$ if necessary*) *such that for every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_{p_1}^+\right)$, there exists a holomorphic function $F \in \mathcal{O}\left(\mathcal{W}_1 \cup \left[\Omega \cup \mathcal{HW}_1^+\right]\right)$ with $F|_{\Omega \cup \mathcal{HW}_{p_1}^+} = f$.*
- **(II)** *If v_1 belongs to $T_{p_1}^c M$, then there exists a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$ with $\omega_{p_1} \cap \left[\Omega \cup \mathcal{HW}_{p_1}^+\right]$ connected* (*shrinking $\Omega \cup \mathcal{HW}_{p_1}^+$ if necessary*) *such that for every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_{p_1}^+\right)$, there exists a holomorphic function $F \in \mathcal{O}\left(\omega_{p_1} \cup \left[\Omega \cup \mathcal{HW}_1^+\right]\right)$ with $F|_{\Omega \cup \mathcal{HW}_{p_1}^+} = f$.*

In the CR dimension $m \geq 2$ case, we observe that an analogous main removability proposition may be formulated simply by adding to the assumptions of Proposition 3.22 a local half-wedge $\mathcal{HW}_{p_1}^+$, whose existence was established in §4.64 above. The remainder of Section 5, and then Section 6, Section 7, Section 8 and Section 9 below will be entirely devoted to the proof of Proposition 5.12.

5.13. A dichotomy. Under the assumptions of Proposition 5.12, we shall indeed distinguish two cases:

(I) The nonzero vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T^c_{p_1}M$.

(II) The nonzero vector v_1 belongs to the characteristic direction $T_{p_1}M^1 \cap T^c_{p_1}M$.

We must clarify the main assumption that v_1 belongs to the filling FC_{p_1} of the cone $\mathsf{C}_{p_1} \subset T_{p_1}M^1$ created by the local half-wedge $\mathcal{HW}^+_{p_1}$. As we have observed in §4.2, in the (generic) situation of Case **(I)**, a local half-wedge may be represented geometrically as the intersection of a (complete) local wedge of edge M at p_1, with a local one-sided neighborhood $(N^1)^+$ of a hypersurface N^1 passing through p_1, which is transversal to M and which satisfies $N^1 \cap M \equiv M^1$ in a neighborhood of p_1. The slope of the tangent space $T_{p_1}N^1$ to N^1 at p_1 with respect to the tangent space $T_{p_1}M$ to M at p_1 may be understood in terms of the cone C_{p_1}, as we will now explain. Afterwards, we shall consider Case **(II)** separately.

5.14. Cones, filled cones, subcones and local description of half-wedges in Case (I). As in some of the assumptions of Proposition 5.12, let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of CR dimension $m = 1$ and of codimension $d = n - 1 \geq 1$, let $p_1 \in M$, let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth one-codimensional submanifold of M passing through p_1. For the sake of concreteness, it will be convenient to work in a holomorphic coordinate system $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ centered at p_1 in which $T_{p_1}M = \{y_2 = \cdots = y_n = 0\}$ and $T_{p_1}M^1 = \{y_1 = y_2 = \cdots = y_n = 0\}$ (the existence of such a coordinate system which straightens both $T_{p_1}M$ and $T_{p_1}M^1$ is a direct consequence of the considerations of §4.11). Let $\pi' : T_{p_1}M^1 \to T_{p_1}M^1/(T_{p_1}M^1 \cap T^c_{p_1}M)$ denote the canonical projection, namey $\pi'(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$. Sometimes, we shall denote the coordinates by $(z_1, z') = (x_1 + iy_1, x' + iy') \in \mathbb{C} \times \mathbb{C}^{n-1}$. In these coordinates, the characteristic direction is given by the x_1-axis and we may assume that the tangent plane at p_1 of the one-sided neighborhood $(M^1)^+$ is given by $T_{p_1}(M^1)^+ = \{y' = 0, \ y_1 > 0\}$.

Let $\mathsf{C}_{p_1} \subset T_{p_1}M^1$ be the infinite open cone created by $\mathcal{HW}^+_{p_1}$ at p_1 and let $\mathsf{FC}_{p_1} \subset T_{p_1}M^1$ be its filling. Let $\mathsf{C}'_{p_1} := \pi'(\mathsf{C}_{p_1})$ be its projection onto the x'-space, which yields an $(n-1)$-dimensional infinite cone in the x'-space, open with respect to its topology. Notice that, by the definition (5.6) of the filling (along the characteristic direction) of a cone, the two projections $\pi'(\mathsf{C}_{p_1})$ and $\pi'(\mathsf{FC}_{p_1})$ are identical. We now need to explain how these three cones C_{p_1}, FC_{p_1}, C'_{p_1} and the nonzero vector $v_1 \in \mathsf{FC}_{p_1}$ are disposed, geometrically, *see* FIGURE 12 just below.

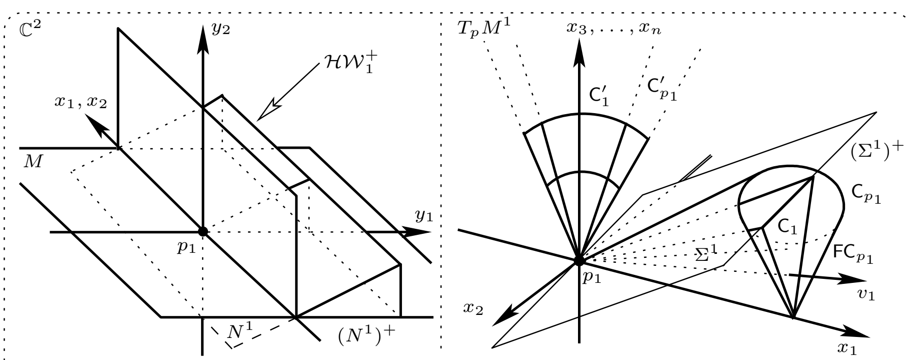

FIGURE 12: CONE CREATED BY THE HALF-WEDGE AND ITS RELATION TO THE HALF-WEDGE

Because the disc $\mathcal{Z}_{t,\chi,\nu}$ of Proposition 5.12 (which is a localization in a neighborhood of the special point of the discs constructed in Section 4) is small, the tangent vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ is necessarily close to the complex tangent plane $T^c_{p_1}M$: this may be checked directly by differentiating Bishop's equation (4.40) with respect to θ, using the fact that the $\mathcal{C}^1$-norm of Φ' is small. Moreover, since this vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$ also belongs to $T_{p_1}M^1$, it is in fact close to the positive x_1-axis. Furthermore, since the vector v_1 belongs to the filling of the open cone C_{p_1} which contains the vector $\frac{\partial \mathcal{Z}_{0,0,0}}{\partial \theta}(1)$, and since in the proof of Lemma 5.9 above we have chosen the special point, the supporting hypersurface and the vector v_1 with a parameter λ very close to 1, so that the vector field $p \mapsto \widetilde{v}_p^\lambda$ was very close to the characteristic vector field $p \mapsto X_p$, it follows that the vector $v_1 \equiv \widetilde{v}^\lambda_{p_{\rm sp}}$ is even closer to the positive x_1-axis. However, we suppose in Case **(I)** that v_1 is *not* directed along the x_1-axis, so v_1 has coordinates $(v_{1;1}, v_{2;1}, \dots, v_{n;1}) \in \mathbb{R}^n$ with $v_{1;1} > 0$, with $|v_{j;1}| << v_{1;1}$ for $j = 2, \dots, n$ and with at least one $v_{j;1}$ being nonzero.

We need some general terminology. Let C be an open infinite cone in a real linear subspace E of dimension $q \geq 1$. We say that C' is a *proper subcone* and we write $\mathsf{C}' \subset\subset \mathsf{C}$ (*see* the left hand side of FIGURE 10 above for an illustration) if the intersection of C' with the unit sphere of E is a relatively compact subset of the intersection of C with the unit sphere of E, this property being independent of the choice of a norm on E. We say that C is a *linear cone* if it may be defined by $\mathsf{C} = \{x \in E : \ \ell_1(x) > 0, \dots, \ell_q(x) > 0\}$ for some q linearly independent real linear forms $\ell_1, \dots, \ell_q$ on E.

In the $(x_2, \dots, x_n)$-space, we now choose an open infinite strictly convex linear proper subcone $\mathsf{C}_1' \subset\subset \mathsf{C}_{p_1}'$ with the property that v_1 belongs to its filling FC_1', *cf.* FIGURE 12 above. Here, we may assume that C_1' is described by $(n-1)$ strict inequalities $\ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0$, where the $\ell_j'(x')$ are linearly independent linear forms. It then follows that there exists a linear form $\sigma(x_1, x')$ of the form $\sigma(x_1, x') = x_1 + a_2 x_2 + \cdots + a_n x_n$ such that the original filled cone FC_{p_1} is contained in the linear cone

$$\mathsf{C}_1 := \left\{(x_1, x') \in \mathbb{R}^n : \ \ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0, \ \sigma(x_1, x') > 0\right\}, \tag{5.15}$$

which contains the vector v_1. This cone is automatically filled, namely $\mathsf{C}_1 \equiv \mathsf{FC}_1$.

We remind that by genericity of M, the complex structure J of $T\mathbb{C}^n$ induces an isomorphism $T_{p_1}M / T^c_{p_1}M \to T_{p_1}\mathbb{C}^n / T_{p_1}M$. Hence $J\mathsf{C}_{p_1}'$ and $J\mathsf{C}_1'$ are open infinite strictly convex linear proper cones in $T_{p_1}\mathbb{C}^n / T_{p_1}M \cong \{(0, y') \in \mathbb{C}^n\}$. Since $J\mathsf{C}_1'$ is a proper subcone of $J\mathsf{C}_{p_1}'$ and since in the classical definition of a wedge, only the projection of the cone on the quotient space $T_{p_1}M / T^c_{p_1}M$ has a contribution to the wedge, it then follows that the complete wedge $\mathcal{W}_{p_1}$ associated to the family $\mathcal{Z}_{t,\chi,\nu}(\zeta)$ (*cf.* the paragraph after (4.5)) contains a wedge of the form

$$\mathcal{W}_1 := \{p + \mathsf{c}_1' : \ p \in M, \ \mathsf{c}_1' \in J\mathsf{C}_1'\} \cap \Delta_n(p_1, \delta_1), \tag{5.16}$$

for some δ_1 with $0 < \delta_1 < \varepsilon$, where ε is as in §4.2. Furthermore, as observed in §4.2, there exists a $\mathcal{C}^{2,\alpha}$-smooth hypersurface N^1 of $\mathbb{C}^n$ passing through p_1 with the property that $N^1 \cap M \equiv M^1$ locally in a neighborhood of p_1 such that, shrinking $\delta_1 > 0$ if necessary, the local half-wedge $\mathcal{HW}^+_{p_1}$ contains a local half-wedge $\mathcal{HW}^+_1$ of edge $(M^1)^+$ at p_1 which is described as the geometric intersection of the complete wedge $\mathcal{W}_{p_1}$ with a one-sided neighborhood $(N^1)^+$, namely

$$\mathcal{HW}^+_1 := \mathcal{W}_1 \cap (N^1)^+. \tag{5.17}$$

An illustration for the case $n = 2$ where $M \subset \mathbb{C}^2$ is a hypesurface is provided in the left hand side of FIGURE 12. In addition, it follows from the definition of $\mathcal{HW}^+_{p_1}$ by means

of the segments $\mathcal{Z}_{t,\chi,\nu}\left((1-\varepsilon,1)\right)$ that we can assume that

$$T_{p_1}(N^1)^+ = T_{p_1}M \oplus J(\Sigma^1)^+, \tag{5.18}$$

where $(\Sigma_1)^+$ is the hyperplane one-sided neighborhood $\{(x_1,x') : \ \sigma(x_1,x') > 0\} \subset T_{p_1}M^1$. Equivalently, $T_{p_1}(N^1)^+$ is represented by the inequality $y_1 + a_2y_2 + \cdots + a_ny_n > 0$. Consequently, there exists a $\mathcal{C}^{2,\alpha}$-smooth function $\psi(x,y')$ with $\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0$ for $k = 1,\dots,n$ and $j = 2,\dots,n$ such that N^1 is represented by the equation $y_1 + a_2y_2 + \cdots + a_ny_n = \psi(x,y')$ and $(N^1)^+$ by the inequation $y_1 + a_2y_2 + \cdots + a_ny_n > \psi(x,y')$.

5.19. Cones, filled cones, subcones and local description of half-wedges in Case (II). Secondly, we assume that the nonzero vector v_1 of Proposition 5.12 belongs to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM$. In this case, as observed in §4.2, the half-wedge $\mathcal{HW}_{p_1}^+$ coincides with a local wedge of edge M^1 at (p_1, Jv_1). After a real dilatation of the z_1-axis, we can assume that $v_1 = (1,0,\dots,0)$. Choosing an open infinite strictly convex linear proper subcone $\mathsf{C}_2 \subset\subset \mathsf{C}_{p_1} \subset T_{p_1}M^1 = \mathbb{R}_x^n$ defined by n strict inequalities $\ell_1(x) > 0,\dots,\ell_n(x) > 0$, where the $\ell_j(x)$ are linearly independent real linear forms – of course with C_2 containing the vector v_1 – it follows that there exists $\delta_1 > 0$ such that the half-wedge $\mathcal{HW}_{p_1}^+$ contains the following local wedge of edge M^1 at p_1:

$$\mathcal{W}_2 := \left\{p + \mathsf{c}_2 : \ p \in M^1, \ \mathsf{c}_2 \in J\mathsf{C}_2\right\} \cap \Delta_n(p_1,\delta_1). \tag{5.20}$$

We remind that it was observed in §4.2 (*cf.* especially the right hand side of FIGURE 5) that $\mathcal{W}_2$ contains $(M^1)^+$ locally in a neighborhood of p_1. In §5.22 below, we shall provide a more concrete representation of $\mathcal{W}_2$ in an appropriate system of coordinates.

5.21. A trichotomy. Let us pursue this discussion more concretely by introducing further normalizations. Our goal will now be to construct appropriate normalized coordinate systems. Analyzing further the dichotomy introduced in §5.13 by taking account of the presence of the one-codimensional submanifold $H^1 \subset M^1$, we shall distinguish three cases by dividing Case **(I)** in two subcases **(I$_1$)** and **(I$_2$)** as follows:

- **(I$_1$)** The nonzero vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM$ and $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T_{p_1}^cM\right) = 0$.
- **(I$_2$)** The nonzero vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM$ and $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T_{p_1}^cM\right) = 1$ (this possibility can only occur when $n \geq 3$).
- **(II)** The nonzero vector v_1 belongs to the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM$.

In case **(I$_1$)**, we notice that the assumption $T_{p_1}H^1 \cap T_{p_1}^cM = \{0\}$ implies that v_1 does not belong to the characteristic direction, because $v_1 \in T_{p_1}H^1$. Also, in case **(II)**, we notice that $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T_{p_1}^cM\right) = 1$ because $v_1 \in T_{p_1}H^1$, because $T_{p_1}H^1 \subset T_{p_1}M^1$ and because the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM$ is one-dimensional.

In each of the above three cases, it will be convenient in Section 8 below to work with simultaneously normalized defining (in)equations for M, for M^1, for $(M^1)^+$, for H^1, for $(H^1)^+$, for C_1', for v_1, for C_1, for $(N^1)^+$ and for $\mathcal{HW}_{p_1}^+$, in a single coordinate system centered at p_1. In the next paragraphs, we shall set up further elementary normalization lemmas in a *common system of coordinates*, firstly for Case **(I$_1$)**, secondly for Case **(I$_2$)** and thirdly for Case **(II)**.

First of all, in the above coordinate system (z_1,z') with $T_{p_1}M = \{y_2 = \cdots = y_n = 0\}$ and with $T_{p_1}M^1 = \{y_1 = y_2 = \cdots = y_n = 0\}$, by means of the implicit function theorem, we can represent locally M by $(n-1)$ grahed equations of the form

$y_2 = \varphi_2(x, y_1), \dots, y_n = \varphi_n(x, y_1)$, where the φ_j are $\mathcal{C}^{2,\alpha}$-smooth functions satisfying $\varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_1}\varphi_j(0) = 0$ for $j = 2, \dots, n$, $k = 1, \dots, n$ and we can represent M^1 by n graphed equations $y_1 = h_1(x), y_2 = h_2(x), \dots, y_n = h_n(x)$, where the h_j are $\mathcal{C}^{2,\alpha}$-smooth functions satisfying $h_j(0) = \partial_{x_k}h_j(0) = 0$ for $j, k = 1, \dots, n$.

5.22. First order normalizations in Case ($\mathbf{I_1}$). Thus, let us deal first with Case **($\mathbf{I_1}$)**. After a possible permutation of coordinates, we can assume that $T_{p_1}H^1$, which is a one-codimensional subspace of $T_{p_1}M^1$, is given by the equations

$$x_1 = b_2x_2 + \cdots + b_nx_n,\ y_1 = 0, y' = 0, \tag{5.23}$$

for some real numbers $b_2, \dots, b_n$. If we define the linear invertible transformation $\widehat{z}_1 := z_1 - b_2z_2 - \cdots - b_nz_n$, $\widehat{z}' := z'$, then the plane $T_{p_1}H^1$ written in (5.23) clearly transforms to the plane $\widehat{x}_1 = \widehat{y}_1 = \widehat{y}' = 0$, and (fortunately) $T_{p_1}M$ and $T_{p_1}M^1$ are left unchanged, namely $T_{p_1}\widehat{M} = \{\widehat{y}' = 0\}$ and $T_{p_1}\widehat{M^1} = \{\widehat{y}_1 = \widehat{y}' = 0\}$.

Dropping the hats on coordinates, we have $T_{p_1}M = \{y' = 0\}$, $T_{p_1}M^1 = \{y_1 = y' = 0\}$, $T_{p_1}H^1 = \{x_1 = y_1 = y' = 0\}$. Let $\mathsf{C}_1' \subset\subset \mathsf{C}_{p_1}'$ be the open infinite strictly convex linear cone introduced in §5.14, which is contained in the real $(n-1)$-dimensional space $\{(0, x')\}$ and which is defined by $(n-1)$ strict inequalities $\ell_1'(x') > 0, \dots, \ell_{n-1}'(x') > 0$. By means of a real linear invertible transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' := A' \cdot z'$, where A' is an $(n-1) \times (n-1)$ real matrix, we can transform C_1' to a cone $\widehat{\mathsf{C}}_1'$ defined by the simpler inequalities $\widehat{x}_2 > 0, \dots, \widehat{x}_n > 0$. Fortunately, this transformation stabilizes $T_{p_1}M$, $T_{p_1}M^1$ and $T_{p_1}H^1$.

Dropping the hats on coordinates, we now have $T_{p_1}M = \{y' = 0\}$, $T_{p_1}M^1 = \{y_1 = y' = 0\}$, $T_{p_1}H^1 = \{x_1 = y_1 = y' = 0\}$ and $\mathsf{C}_1' = \{(0, x') :\ x_2 > 0, \dots, x_n > 0\}$. Then the nonzero vector $v_1 \in T_{p_1}H^1$ which belongs to C_1' has coordinates $v_1 = (0, v_{2;1}, \dots, v_{n;1}) \in \mathbb{R}^n$, where $v_{2;1} > 0, \dots, v_{n;1} > 0$. By means of real dilatations or real contractions of the real axes $\mathbb{R}_{x_2}, \dots, \mathbb{R}_{x_n}$ (a transformation which does not perturb the previously achieved normalizations), we can assume that $v_1 = (0, 1, \dots, 1)$ and that $T_{p_1}(M^1)^+ = \{y' = 0, y_1 > 0\}$, $T_{p_1}(H^1)^+ = \{y = 0,\ x_1 > 0\}$.

Finally, the linear one-codimensional subspace $\sigma_1 \subset T_{p_1}M^1$ introduced in §5.14 which does not contain the characteristic direction $T_{p_1}M^1 \cap T_{p_1}^cM \equiv \mathbb{R}_{x_1}$ may be represented by an equation of the form $\sigma(x_1, x') := x_1 + a_2x_2 + \cdots + a_nx_n = 0$, for some real numbers $a_2, \dots, a_n$. By (5.15), the vector v_1 belongs to the cone C_1, hence $a_2 + \cdots + a_n > 0$. After a dilatation of the x_1-axis, we can even assume that $a_2 + \cdots + a_n = 1$. We remind that by (5.18), the half-space $T_{p_1}(N^1)^+$ is given by $y_1 + a_2y_2 + \cdots + a_ny_n > 0$, hence there exists a $\mathcal{C}^{2,\alpha}$-smooth function $\psi(x, y')$ with $\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0$ for $k = 1, \dots, n$ and $j = 2, \dots, n$ such that $(N^1)^+$ is represented by the inequation $y_1 + a_2y_2 + \cdots + a_ny_n > \psi(x, y')$. Consequently, in this coordinate system, we may represent concretely the local half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ constructed in §5.14 as

$$\begin{cases} \mathcal{HW}_1^+ = \{(z_1, z') \in \mathbb{C}^n : |z_1| < \delta_1,\ |z'| < \delta_1, \\ \qquad\qquad y_1 + a_2y_2 + \cdots + a_ny_n - \psi(x, y') > 0, \\ \qquad\qquad y_2 - \varphi_2(x, y_1) > 0, \dots, y_n - \varphi_n(x, y_1) > 0\ \}. \end{cases} \tag{5.24}$$

For the continuation of the proof of Proposition 5.12, it will also be convenient to proceed to further second order normalizations of the totally real submanifolds M^1 and H^1. These normalizations will all be tangent to the identity tranformation, hence they will leave the previously achieved normalizations unchanged.

5.25. Second order normalizations in Case (I_1). Let us then perform a second order Taylor development of the defining equations of M^1

$$y = h(x) = \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, x_{k_1} x_{k_2} + o(|x|^2), \tag{5.26}$$

where the $a_{k_1,k_2} = \frac{1}{2}\partial_{x_{k_1}}\partial_{x_{k_2}} h(0)$ are vectors of $\mathbb{R}^n$. If we define the quadratic invertible transformation

$$\widehat{z} := z - i \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, z_{k_1} z_{k_2} = \Phi(z), \tag{5.27}$$

which is tangent to the identity mapping at the origin, then for $x+iy = x+ih(x) \in M^1$, we have by replacing (5.26) in the imaginary part of $\widehat{z}$ given by (5.27)

$$\begin{cases} \widehat{y} = y - \displaystyle\sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, x_{k_1} x_{k_2} + \sum_{k_1,k_2=1}^{n} a_{k_1,k_2}\, y_{k_1} y_{k_2} \\ \quad = \mathrm{o}(|x|^2) \\ \quad = \mathrm{o}\left(\left|\mathrm{Re}\,\Phi^{-1}(\widehat{z})\right|^2\right) = \mathrm{o}\left(|(\widehat{x},\widehat{y})|^2\right), \end{cases} \tag{5.28}$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve (5.28) in terms of $\widehat{y}$, we find that $\widehat{M^1} := \Phi(M^1)$ may be represented by an equation of the form $\widehat{y} = \widehat{h}(\widehat{x})$, for some $\mathbb{R}^n$-valued local $\mathcal{C}^{2,\alpha}$-smooth mapping $\widehat{h}$ which satisfies $\widehat{h}(\widehat{x}) = \mathrm{o}(|\widehat{x}|^2)$.

Finally, dropping the hats on coordinates, we can assume that the functions $h_1, \dots, h_n$ vanish at the origin to second order. Since $T_{p_1}H^1 = \{y = 0,\ x_1 = 0\}$, there exists a $\mathcal{C}^{2,\alpha}$-smooth function $g(x')$ with $g(0) = \partial_{x_k} g(0) = 0$ for $k = 2, \dots, n$ such that $(H^1)^+$ is given by the equation $x_1 > g(x')$. We want to normalize also the defining equation $x_1 = g(x')$ of H^1. Instead of requiring, similarly as for $h_1, \dots, h_n$, that g vanishes to second order at the origin (which would be possible), we shall normalize g in order that $g(x') = -x_1^2 - \cdots - x_n^2 + \mathrm{o}\left(|x'|^{2+\alpha}\right)$ (which will also be possible, thanks to the total reality of H^1). The reason why we want g to be strictly concave is a trick that will be useful in Section 8 below.

Thus, we now perform a second order Taylor development of the defining equations of H^1

$$\begin{cases} x_1 = g\left(x'\right) = \displaystyle\sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, x_{k_1} x_{k_2} + o(|x'|^2), \\ y = h\left(g(x'), x'\right) =: k\left(x'\right) = \mathrm{o}(|x'|^2), \end{cases} \tag{5.29}$$

where the $b_{k_1,k_2} = \frac{1}{2}\partial_{x_{k_1}}\partial_{x_{k_2}} g(0)$ are real numbers. If we define the quadratic invertible transformation

$$\begin{cases} \widehat{z}_1 := z_1 - \displaystyle\sum_{k_1,k_2=2}^{n} b_{k_1,k_2}\, z_{k_1} z_{k_2} - z_2^2 - \cdots - z_n^2, \\ \widehat{z}' := z', \end{cases} \tag{5.30}$$

which is tangent to the identity mapping, then for $(g(x')+ik_1(x'),x'+ik'(x')) \in H^1$, we have by replacing (5.29) in the real part of $\widehat{z}_1$, given by (5.30):

(5.31)
$$\begin{cases} \widehat{x}_1 = x_1 - \sum_{k_1,k_2=2}^n b_{k_1,k_2}\, x_{k_1}x_{k_2} + \sum_{k_1,k_2=2}^n b_{k_1,k_2}\, y_{k_1}y_{k_2} - \sum_{k=2}^n x_k^2 + \sum_{k=2}^n y_k^2, \\ \quad = \, -x_2^2 - \cdots - x_n^2 + \mathrm{o}\left(|x'|^2\right) \\ \quad = \, -\widehat{x}_2^2 - \cdots - \widehat{x}_n^2 - \mathrm{o}\left(|(\widehat{x},\widehat{y})|^2\right). \end{cases}$$

Similarly (dropping the elementary computations), we may obtain for the imaginary part of $\widehat{z}_1$ and for the imaginary part of $\widehat{z}'$

$$\widehat{y}_1 = \mathrm{o}\left(|(\widehat{x},\widehat{y})|^2\right) \quad \text{and} \quad \widehat{y}' = \mathrm{o}\left(|(\widehat{x},\widehat{y})|^2\right), \tag{5.32}$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve the system (5.31), (5.32) in terms of $\widehat{x}_1$, $\widehat{y}_1$ and $\widehat{y}'$, we find that $\widehat{H}^1 := \Phi(H^1)$ may be represented by equations of the form

$$\begin{cases} \widehat{x}_1 = \widehat{g}\,(\widehat{x}') = -\widehat{x}_2^2 - \cdots - \widehat{x}_n^2 + \mathrm{o}\left(|\widehat{x}'|^2\right), \\ \widehat{y} = \widehat{k}\,(\widehat{x}') = o\left(|\widehat{x}'|^2\right). \end{cases} \tag{5.33}$$

It remains to check that the above transformation has not perturbed the previous second order normalizations of $h_1,\dots,h_n$ (this is important), which is easy: replacing y by $h(x) = \mathrm{o}(|x|^2)$ in the imaginary parts of $\widehat{z}_1$ and of $\widehat{z}'$ defined by the transformation (5.30), we get firstly

$$\begin{cases} \widehat{y}_1 = y_1 - \sum_{k_1,k_2}^n b_{k_1,k_2}\, (x_{k_1}y_{k_2} + y_{k_1}x_{k_2}) - 2\sum_{k=2}^n x_ky_k \\ \quad = \mathrm{o}(|x|^2) \\ \quad = \mathrm{o}\left(\left|\mathrm{Re}\,\Phi^{-1}(\widehat{z})\right|^2\right) = \mathrm{o}\left(|(\widehat{x},\widehat{y})|^2\right), \end{cases} \tag{5.34}$$

and similarly

$$\widehat{y}' = o\left(|(\widehat{x},\widehat{y})|^2\right), \tag{5.35}$$

whence by applying the $\mathcal{C}^{2,\alpha}$ implicit function theorem to solve the system (5.34), (5.35) in terms of $\widehat{y}$, we find that $\widehat{M}^1 := \Phi(M^1)$ may be represented by equations of the form $\widehat{y} = \widehat{h}\,(\widehat{x}) = \mathrm{o}\left(|\widehat{x}|^2\right)$. Thus, after dropping the hats on coordinates, all the desired normalizations are satisfied. We shall now summarize these normalizations and we shall formulate just afterwards the analogous normalizations for Cases $\mathbf{(I_2)}$ and **(II)**.

5.36. Simultaneous normalization lemma. In the following lemma, the final choice of sufficiently small radii $\rho_1 > 0$ and $\delta_1 > 0$ is made after that all the biholomorphic changes of coordinates and all the applications of the implicit function theorem are achieved.

Lemma 5.37. *Let M, M^1, p_1, H^1, $(H^1)^+$, $\mathcal{HW}_{p_1}^+$, C_{p_1} and FC_{p_1} be as in Proposition 5.12. Then there exists a sub-half-wedge $\mathcal{HW}_1^+$ contained in $\mathcal{HW}_{p_1}^+$ such that the following normalizations hold in each of the three cases* $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ *and* **(II)**:

(I$_1$) *If* $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T_{p_1}^c M\right) = 0$*, then there exists a system of holomorphic coordinates* $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ *vanishing at* p_1 *with the vector* v_1 *equal to* $(0, 1, \dots, 1)$*, there exists positive numbers* ρ_1 *and* δ_1 *with* $0 < \delta_1 < \rho_1$*, there exist* $\mathcal{C}^{2,\alpha}$*-smooth functions* $\varphi_2, \dots, \varphi_n$*,* $h_1, \dots, h_n$*,* g*,* $k_1, \dots, k_n, \psi$*, all defined in real cubes of edge* $2\rho_1$ *and of the appropriate dimension, and there exist real numbers* $a_1, \dots, a_n$ *with* $a_2 + \cdots + a_n = 1$*, such that, if we denote* $z' := (z_2, \dots, z_n) = x' + iy'$*, then* M*,* M^1*,* $(M^1)^+$*,* H^1*,* $(H^1)^+$ *and* N^1 *are represented in the polydisc of radius* ρ_1 *centered at* p_1 *by the following graphed (in)equations and the sub-half-wedge* $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ *is represented in the polydisc of radius* δ_1 *centered at* p_1 *by the following inequations*

$$
\tag{5.38}
\left\{
\begin{aligned}
M: \quad & y_2 = \varphi_2(x, y_1), \dots\dots,\ y_n = \varphi_n(x, y_1),\\
M^1: \quad & y_1 = h_1(x), y_2 = h_2(x), \dots\dots,\ y_n = h_n(x),\\
(M^1)^+: \quad & y_1 > h_1(x), y_2 = \varphi_2(x, y_1), \dots\dots,\ y_n = \varphi_n(x, y_1),\\
H^1: \quad & x_1 = g(x'),\ y_1 = k_1(x'), \dots\dots,\ y_n = k_n(x'),\\
(H^1)^+: \quad & x_1 > g(x'),\ y_1 = h_1(x), y_2 = h_2(x), \dots\dots,\ y_n = h_n(x),\\
N^1: \quad & y_1 + a_2y_2 + \cdots + a_ny_n = \psi(x, y'),\\
\mathcal{HW}_1^+: \quad & y_1 + a_2y_2 + \cdots + a_ny_n > \psi(x, y'),\\
& \qquad\qquad y_2 > \varphi_2(x, y_1), \dots, y_n > \varphi_n(x, y_1),
\end{aligned}
\right.
$$

where we can assume that M^1 *coincides with the intersection* $M \cap \{y_1 = h_1(x)\}$*, that* H^1 *coincides with the intersection* $M^1 \cap \{x_1 = g(x')\}$ *and that* N^1 *contains* M^1*, which yields at the level of defining equations the following three collections of identities*

$$
\tag{5.39}
\left\{
\begin{aligned}
h_2(x) \equiv \varphi_2(x, h_1(x)), \dots\dots, h_n(x) \equiv \varphi_n(x, h_1(x)),\\
k_1(x') \equiv h_1(g(x'), x'), \dots\dots, k_n(x') \equiv h_n(g(x'), x'),\\
\psi(x, h'(x)) \equiv h_1(x) + a_2h_2(x) + \cdots + a_nh_n(x),
\end{aligned}
\right.
$$

and where the following normalizations hold (where δ_a^b*, equal to* 0 *if* $a \neq b$ *and to* 1 *if* $a = b$*, denotes Kronecker's symbol)*:

$$
\tag{5.40}
\left\{
\begin{aligned}
&\varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_1}\varphi_j(0) = 0, \quad j = 2, \dots, n,\ k = 1, \dots, n,\\
&h_j(0) = \partial_{x_k}h_j(0) = \partial_{x_{k_1}}\partial_{x_{k_2}}h_j(0) = 0, \qquad j, k, k_1, k_2 = 1, \dots, n,\\
&g(0) = \partial_{x_k}g(0) = k_j(0) = \partial_{x_k}k_j(0) = 0, \quad j = 1, \dots, n,\ k = 2, \dots, n,\\
&\qquad\qquad\qquad\qquad \partial_{x_{k_1}}\partial_{x_{k_2}}g(0) = -\delta_{k_1}^{k_2}, \qquad k_1, k_2 = 2, \dots, n,\\
&\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0, \quad k = 1, \dots, n,\ j = 2, \dots, n.
\end{aligned}
\right.
$$

In other words, $T_0M = \{y' = 0\}$ *(hence* T_0^cM *coincides with the complex* z_1*-axis),* $T_0N^1 = \{y_1 + a_2y_2 + \cdots + a_ny_n = 0\}$ *and the second order Taylor approximations of the defining equations of* M^1*, of* H^1 *and of* $(H^1)^+$ *are the quadrics*

$$
\tag{5.41}
\left\{
\begin{aligned}
T_{p_1}^{(2)}M^1: \quad & y_1 = 0, \dots\dots,\ y_n = 0,\\
T_{p_1}^{(2)}H^1: \quad & x_1 = -x_2^2 - \cdots - x_n^2,\ y_1 = 0, \dots\dots,\ y_n = 0,\\
T_{p_1}^{(2)}(H^1)^+: \quad & x_1 > -x_2^2 - \cdots - x_n^2,\ y_1 = 0, \dots\dots,\ y_n = 0.
\end{aligned}
\right.
$$

(I₂) *Similarly, if* $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T_{p_1}^c M\right) = 1$ *and if* v_1 *is not complex tangent to* M (*this possibility can only occur in the case* $n \geq 3$), *then there exists a system of holomorphic coordinates* $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ *vanishing at* p_1 *with* v_1 *equal to* $(1, \dots, 1, 0)$, *there exists positive numbers* ρ_1 *and* δ_1 *with* $0 < \delta_1 < \rho_1$, *there exist* $\mathcal{C}^{2,\alpha}$*-smoooth functions* $\varphi_2, \dots, \varphi_n, h_1, \dots, h_n, g, k_1, \dots, k_n, \psi$ *all defined in real cubes of edge* $2\rho_1$ *and of the appropriate dimension, such that if we denote* $z'' := (z_1, \dots, z_{n-1}) = x'' + iy''$ *and* $z' = (z_2, \dots, z_n) = x' + iy'$, *then* M, M^1, $(M^1)^+$, H^1, $(H^1)^+$ *and* N^1 *are represented in the polydisc of radius* ρ_1 *centered at* p_1 *by the following graphed (in)equations and the sub-half-wedge* $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ *is represented in the polydisc of radius* δ_1 *centered at* p_1 *by the following inequations*

$$
\left\{
\begin{aligned}
M: \quad & y_2 = \varphi_2(x, y_1), \dots\dots, \ y_n = \varphi_n(x, y_1), \\
M^1: \quad & y_1 = h_1(x), y_2 = h_2(x), \dots\dots, \ y_n = h_n(x), \\
(M^1)^+: \quad & y_1 > h_1(x), y_2 = \varphi_2(x, y_1), \dots\dots, \ y_n = \varphi_n(x, y_1), \\
H^1: \quad & x_n = g(x''), \ y_1 = k_1(x''), \dots\dots, \ y_n = k_n(x''), \\
(H^1)^+: \quad & x_n > g(x''), \ y_1 = h_1(x), y_2 = h_2(x), \dots\dots, \ y_n = h_n(x), \\
N^1: \quad & y_2 + \cdots + y_{n-1} - y_n = \psi(x, y'), \\
\mathcal{HW}_1^+: \quad & y_2 + \cdots + y_{n-1} - y_n > \psi(x, y'), \\
& \qquad y_1 > \varphi_1(x, y_1), \dots, y_{n-1} > \varphi_{n-1}(x, y_1),
\end{aligned}
\right.
\tag{5.42}
$$

where we can assume that M^1 *coincides with the intersection* $M \cap \{y_1 = h_1(x)\}$, *that* H^1 *coincides with the intersection* $M^1 \cap \{x_1 = g(x')\}$ *and that* N^1 *contains* M^1, *which yields at the level of defining equations the following three collections of identities*

$$
\left\{
\begin{aligned}
h_2(x) &\equiv \varphi_2(x, h_1(x)), \dots\dots, h_n(x) \equiv \varphi_n(x, h_1(x)), \\
k_1(x'') &\equiv h_1(x'', g(x'')), \dots\dots, k_n(x'') \equiv h_n(x'', g(x'')), \\
\psi(x, h'(x)) &\equiv h_1(x) + h_2(x) + \cdots + h_{n-1}(x) - h_n(x),
\end{aligned}
\right.
\tag{5.43}
$$

and where the following normalizations hold:

$$
\left\{
\begin{aligned}
&\varphi_j(0) = \partial_{x_k}\varphi_j(0) = \partial_{y_1}\varphi_j(0) = 0, \quad j = 2, \dots, n, \ k = 2, \dots, n, \\
&h_j(0) = \partial_{x_k}h_j(0) = \partial_{x_{k_1}}\partial_{x_{k_2}}h_j(0) = 0, \qquad j, k, k_1, k_2 = 1, \dots, n, \\
&g(0) = \partial_{x_k}g(0) = k_j(0) = \partial_{x_k}k_j(0) = 0, \quad j = 1, \dots, n, \ k = 1, \dots, n-1, \\
&\qquad\qquad\qquad \partial_{x_{k_1}}\partial_{x_{k_2}}g(0) = -\delta_{k_1}^{k_2}, \qquad k_1, k_2 = 1, \dots, n-1, \\
&\psi(0) = \partial_{x_k}\psi(0) = \partial_{y_j}\psi(0) = 0, \quad k = 1, \dots, n, \ j = 2, \dots, n.
\end{aligned}
\right.
\tag{5.44}
$$

In other words, $T_0M = \{y' = 0\}$ (*hence* T_0^cM *coincides with the complex* z_1*-axis*), $T_0N^1 = \{y_1 + y_2 + \cdots + y_{n-1} - y_n = 0\}$ *and the second order Taylor approximations of the defining equations of* M^1, *of* H^1 *and of* $(H^1)^+$ *are the quadrics*

$$
\left\{
\begin{aligned}
T_{p_1}^{(2)}M^1: \quad & y_1 = 0, \dots\dots, \ y_n = 0, \\
T_{p_1}^{(2)}H^1: \quad & x_n = -x_1^2 - \cdots - x_{n-1}^2, \ y_1 = 0, \dots\dots, \ y_n = 0, \\
T_{p_1}^{(2)}(H^1)^+: \quad & x_n > -x_1^2 - \cdots - x_{n-1}^2, \ y_1 = 0, \dots\dots, \ y_n = 0.
\end{aligned}
\right.
\tag{5.45}
$$

(II) *Finally, if* $\dim_{\mathbb{R}}\left(T_{p_1}H^1 \cap T^c_{p_1}M\right) = 1$ *and if* v_1 *is complex tangent to* M (*this possibility can occur in all cases* $n \geq 2$), *then there exists a system of holomorphic coordinates* $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ *vanishing at* p_1 *with* v_1 *equal to* $(1, 0, \dots, 0)$, *there exist positive numbers* ρ_1 *and* δ_1 *with* $0 < \delta_1 < \rho_1$, *there exist* $\mathcal{C}^{2,\alpha}$*-smoooth functions* $\varphi_2, \dots, \varphi_n, h_1, \dots, h_n, g, k_1, \dots, k_n$ *all defined in real cubes of edge* $2\rho_1$ *and of the appropriate dimension, such that if we denote* $z'' := (z_1, \dots, z_{n-1}) = x'' + iy''$ *and* $z' = (z_2, \dots, z_n) = x' + iy'$, *then* M, M^1, $(M^1)^+$, H^1 *and* $(H^1)^+$ *are represented in the polydisc of radius* ρ_1 *centered at* p_1 *by the first five (in)equations of* (5.42) *together with the normalizations* (5.45) *and such that the local wedge* $\mathcal{W}_2 \subset \mathcal{HW}^+_{p_1}$ *of edge* M^1 *at* p_1 *is represented in the polydisc of radius* δ_1 *centered at* p_1 *by the following inequations*

(5.46)
$$\begin{cases} \mathcal{W}_2: \quad y_1 - h_1(x) > -\left[y_2 - h_2(x)\right], \dots\dots, \; y_1 - h_1(x) > -\left[y_n - h_n(x)\right], \\ \qquad\qquad y_1 - h_1(x) > y_2 - h_2(x) + \cdots + y_n - h_n(x). \end{cases}$$

5.47. Summarizing figure and proof of Lemma 5.37. As an illustration for this technical lemma, by specifying the value $n = 3$, we have drawn in the following figure the cones C_1 and C_2 together with the vector v_1, the tangent plane $T_{p_1}H^1$ and the hyperplane Σ^1 in the three cases $(\mathbf{I_1})$, $(\mathbf{I_2})$ and **(II)**. In the left part of this figure, the cone C_1 is given by $x_2 > 0$, $x_3 > 0$, $x_1 > -\frac{1}{2}x_2 - \frac{1}{2}x_3$, namely we have chosen the values $a_2 = a_3 = \frac{1}{2}$ for the drawing; in the central part, the cone C_1 is given by $x_1 > 0$, $x_2 > 0$, $x_2 > x_3$; in the right part, the cone C_2 is given by $x_1 > -x_2$, $x_1 > -x_3$, $x_1 > x_2 + x_3$.

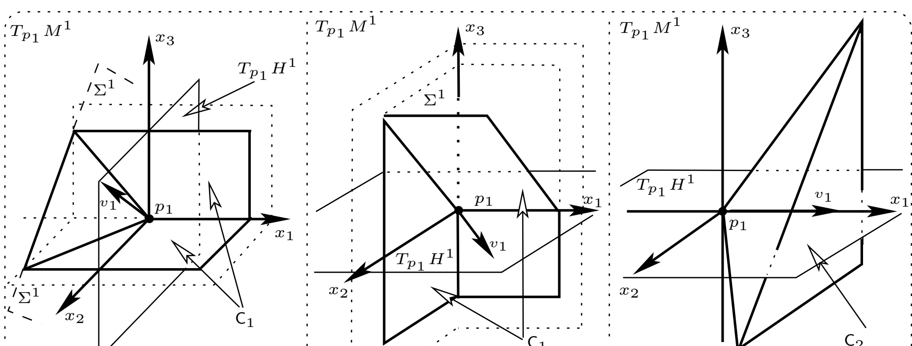

FIGURE 13: THE SUBCONE C_1 IN THE THREE CASES $(\mathbf{I_1})$, $(\mathbf{I_2})$ AND **(II)**

Proof. Case $(\mathbf{I_1})$ has been completed before the statement of Lemma 5.37.

For Case $(\mathbf{I_2})$, we reason similarly, as follows. We start with the normalizations $T_{p_1}M = \{y' = 0\}$ and $T_{p_1}M^1 = \{y = 0\}$ as in the end of §5.21. By assumption, $T_{p_1}H^1$ contains the characteristic direction, which coincides with the x_1-axis. By means of an elementary real linear transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' = A' \cdot z'$, we may first normalize $T_{p_1}H^1$ to be the hyperplane (after dropping the hats on coordinates) $\{x_n = 0, \; y = 0\}$. Similarly, we may normalize v_1 to be the vector $(1, 1, \dots, 1, 0)$. Let again $\pi' : (x_1, x') \mapsto x'$ denote the canonical projection on the x'-space. Then $\pi'(v_1) = (1, \dots, 1, 0)$. Using again a real linear transformation of the form $\widehat{z}_1 := z_1$, $\widehat{z}' = A' \cdot z'$, we can assume that the proper subcone $\mathsf{C}_1' \subset\subset \mathsf{C}'_{p_1} \equiv \pi'(\mathsf{C}_{p_1})$ which

contains the vector v_1 is given (after dropping the hats on coordinates) by

$$\mathsf{C}_1' : \quad x_2 > 0, \dots, x_{n-1} > 0,\ x_2 + \cdots + x_{n-1} > x_n. \tag{5.48}$$

Following §5.14 (*cf.* FIGURE 12), we choose a linear cone $\mathsf{C}_1 \subset\subset \mathsf{FC}_{p_1}$ defined by the $(n-1)$ inequations of C_1' plus one inequation of the form $x_1 > a_2x_2 + \cdots + a_nx_n$ with $1 > a_2 + \cdots + a_{n-1}$, since v_1 belongs to C_1. Then by means of a real linear transformation of the form $\widehat{z}_1 := z_1 + a_2z_2 + \cdots + a_nz_n$, $\widehat{z}' := z'$, which stabilizes $\pi'(v_1)$ and the inequations (5.48) of C_1', we can assume that the supplementary inequation for C_1, namely the inequation for $(\Sigma^1)^+$, is simply (after dropping the hats on coordinates) $x_1 > 0$. Then the vector v_1 is mapped to the vector of coordinates $(1 - a_2 - \cdots - a_n, 1, \dots, 1, 0)$, which we map to the vector of coordinates $(1, 1, \dots, 1, 0)$ by an obvious positive scaling of the x_1-axis. In conclusion, in the final system of coordinates, the cone C_1 is given by

$$\mathsf{C}_1 : \quad x_1 > 0, x_2 > 0, \dots, x_{n-1} > 0,\ x_2 + \cdots + x_{n-1} - x_n > 0. \tag{5.49}$$

This implies that the half-wedge $\mathcal{HW}_1^+ \subset \mathcal{HW}_{p_1}^+$ may be represented by the inequations of the last two line of (5.42). To conclude the proof of Case **($\mathbf{I_2}$)** of Lemma 5.37, it suffices to observe that, as in Case **($\mathbf{I_1}$)**, the further second order normalizations do not perturb the previously achieved first order normalizations, because the transformations are tangent to the identity mapping at the origin.

Finally, we treat Case **(II)** of Lemma 5.37, starting with the system of coordinates $(z_1, \dots, z_n)$ of the end of §5.21. After an elementary real linear transformation stabilizing the characteristic x_1-axis, we can assume that $v_1 = (1, 0, \dots, 0)$ and that the convex infinite linear cone C_2 introduced in §5.19 which contains v_1 is given by the inequations

$$x_1 > -x_2, \dots\dots,\ x_1 > -x_n,\ x_1 > x_2 + \cdots + x_n. \tag{5.50}$$

This implies that the local wedge $\mathcal{W}_2 \subset \mathcal{HW}_{p_1}^+$ of edge M^1 at p_1 introduced in §5.19 may be represented by the inequations (5.46). Finally, the second order normalizations, which are tangent to the identity mapping, are achieved as in the two previous cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**.

The proof of Lemma 5.37 is complete. □

§6. THREE PREPARATORY LEMMAS ON HÖLDER SPACES

In this section, we first collect a few very elementary lemmas that will be useful in our geometric construction of half-attached analytic discs which will be achieved in Section 7 below. From now on, we shall admit the convenient index notation g_{x_k} for the partial derivative which was denoted up to now by $\partial_{x_k} g$.

6.1. Local growth of $\mathcal{C}^{2,\alpha}$-smooth mappings. Let $n \in \mathbb{N}$, $n \geq 1$ and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We shall use the norm $|x| := \max_{1 \leq k \leq n} |x_k|$. If $g = g(x)$ is an $\mathbb{R}^n$-valued $\mathcal{C}^1$-smooth mapping on the real cube $\{x \in \mathbb{R}^n : |x| < 2\rho_1\}$, for some $\rho_1 > 0$, and if $|x'|, |x''| \leq \rho$, for some $\rho < 2\rho_1$, we have the trivial estimate

$$|g(x') - g(x'')| \leq |x' - x''| \cdot \left(\sum_{k=1}^{n} \sup_{|x| \leq \rho} |g_{x_k}(x)| \right), \tag{6.2}$$

where we denote by g_{j,x_k} the partial derivative $\partial g_j / \partial x_k$. Notice that by the definition of the norm $|\cdot|$, we have in (6.2) that $|g(x)| \equiv \max_{1 \leq k \leq n} |g_j(x)|$ and that $|g_{x_k}(x)| \equiv \max_{1 \leq k \leq n} |g_{j,x_k}(x)|$.

Let α with $0 < \alpha < 1$ and let $h = h(x) = (h_1(x), \dots, h_n(x))$ be an $\mathbb{R}^n$-valued mapping which is of class $\mathcal{C}^{2,\alpha}$ on the real cube $\{x \in \mathbb{R}^n : |x| < 2\rho_1\}$, for some $\rho_1 > 0$.

For every $\rho < 2\rho_1$, we consider the $\mathcal{C}^{2,\alpha}$ norm of h over $\{|x| \leq \rho\}$ which is defined precisely as:

(6.3)
$$\left\{ \begin{aligned} ||h||_{\mathcal{C}^{2,\alpha}(\{|x|\leq\rho\})} := \sup_{|x|\leq\rho} |h(x)| + \sum_{k=1}^{n} \sup_{|x|\leq\rho} |h_{x_k}(x)| + \sum_{k_1,k_2=1}^{n} \left|h_{x_{k_1}x_{k_2}}(x)\right| + \\ + \sum_{k_1,k_2=1}^{n} \sup_{|x'|,\,|x''|\leq\rho,\ x'\neq x''} \frac{\left|h_{x_{k_1}x_{k_2}}(x') - h_{x_{k_1}x_{k_2}}(x'')\right|}{|x'-x''|^{\alpha}} < \infty, \end{aligned} \right.$$

and which is finite. With these definitions at hand, the following lemma can easily be established by means of (6.2).

Lemma 6.4. *Under the above assumptions, let*

$$K_1 := ||h||_{\mathcal{C}^{2,\alpha}(\{|x|\leq\rho_1\})} < \infty \tag{6.5}$$

be the $\mathcal{C}^{2,\alpha}$ norm of h over the cube $\{|x| \leq \rho_1\}$ and assume that $h_j(0) = 0$, $h_{j,x_k}(0) = 0$ and $h_{j,x_{k_1}x_{k_2}}(0) = 0$, for all $j, k, k_1, k_2 = 1, \ldots, n$. Then the following three inequalities hold for $|x| \leq \rho_1$:

$$\left\{ \begin{aligned} &[\mathbf{1}]: \quad |h(x)| \leq |x|^{2+\alpha} \cdot K_1, \\ &[\mathbf{2}]: \quad \sum_{k=1}^{n} |h_{x_k}(x)| \leq |x|^{1+\alpha} \cdot K_1, \\ &[\mathbf{3}]: \quad \sum_{k_1,k_2=1}^{n} \left|h_{x_{k_1}x_{k_2}}(x)\right| \leq |x|^{\alpha} \cdot K_1. \end{aligned} \right. \tag{6.6}$$

6.7. A $\mathcal{C}^{1,\alpha}$ estimate for composition of mappings. Recall that Δ is the open unit disc in $\mathbb{C}$ and that $\partial\Delta$ is its boundary, namely the unit circle. We shall constantly denote the complex variable in $\overline{\Delta} := \Delta \cup \partial\Delta$ by $\zeta = \rho\, e^{i\theta}$, where $0 \leq \rho \leq 1$ and where $|\theta| \leq \pi$, except when we consider two points $\zeta' = e^{i\theta'}$, $\zeta'' = e^{i\theta''}$, in which case we may obviously choose $|\theta'|,\ |\theta''| \leq 2\pi$ with $0 \leq |\theta' - \theta''| \leq \pi$. Let now $X(\zeta) = (X_1(\zeta), \ldots, X_n(\zeta))$ be an $\mathbb{R}^n$-valued mapping which is of class $\mathcal{C}^{1,\alpha}$ on the unit circle $\partial\Delta$. We define its $\mathcal{C}^{1,\alpha}$-norm precisely by

$$\left\{ \begin{aligned} ||X||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} := \sup_{|\theta|\leq\pi} \left|X\left(e^{i\theta}\right)\right| + \sup_{|\theta|\leq\pi} \left|\frac{dX\left(e^{i\theta}\right)}{d\theta}\right| + \\ + \sup_{0<|\theta'-\theta''|\leq\pi} \frac{\left|\frac{dX(e^{i\theta'})}{d\theta} - \frac{dX(e^{i\theta''})}{d\theta}\right|}{|\theta'-\theta''|^{\alpha}}, \end{aligned} \right. \tag{6.8}$$

and we define its $\mathcal{C}^1$-norm $||X||_{\mathcal{C}^1(\partial\Delta)}$ by keeping only the first two terms. Let h be as in Lemma 6.5.

Lemma 6.9. *Under the above assumptions, if moreover* $\left|X\left(e^{i\theta}\right)\right| \leq \rho$ *for all* θ *with* $|\theta| \leq \pi$*, where* $\rho \leq \rho_1$*, then we have the following three estimates*:

(6.10)

$$\left\{\begin{aligned}
||h(X)||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq & \sup_{|x|\leq\rho} |h(x)| + \left(\sum_{k=1}^n \sup_{|x|\leq\rho} |h_{x_k}(x)|\right) \cdot ||X||_{\mathcal{C}^1(\partial\Delta)} + \\
& + \left(\sum_{k_1,k_2=1}^n \sup_{|x|\leq\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right|\right) \cdot \pi^{1-\alpha} \cdot \left[||X||_{\mathcal{C}^1(\partial\Delta)}\right]^2 + \\
& + \left(\sum_{k=1}^n \sup_{|x|\leq\rho} |h_{x_k}(x)|\right) \cdot ||X||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}. \\
\sum_{k=1}^n ||h_{x_k}(X)||_{\mathcal{C}^\alpha(\partial\Delta)} \leq & \sum_{k=1} \sup_{|x|\leq\rho} |h_{x_k}(x)| + \\
& + \left(\sum_{k_1,k_2=1}^n \sup_{|x|\leq\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right|\right) \cdot \pi^{1-\alpha} \cdot ||X||_{\mathcal{C}^1(\partial\Delta)}, \\
\sum_{k_1,k_2=1}^n \left|\left|h_{x_{k_1}x_{k_2}}(X)\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} \leq & \sum_{k_1,k_2=1}^n \sup_{|x|\leq\rho} \left|h_{x_{k_1}x_{k_2}}(x)\right| + \\
& + ||h||_{\mathcal{C}^{2,\alpha}(\{|x|\leq\rho\})} \cdot \left(||X||_{\mathcal{C}^1(\partial\Delta)}\right)^\alpha.
\end{aligned}\right.$$

Proof. We summarize the computations. Applying the definition (6.8), using the chain rule for the calculation of $dh\left(X\left(e^{i\theta}\right)\right)/d\theta$, and using the trivial inequality $|a'b' - a''b''| \leq |a'| \cdot |b' - b''| + |b''| \cdot |a' - a''|$, we may majorize

(6.11)

$$\left\{\begin{aligned}
||h(X)||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq & \sup_{|\theta|\leq\pi} \left|h(X(e^{i\theta}))\right| + \left(\sum_{k=1}^n \sup_{|\theta|\leq\pi} \left|h_{x_k}(X(e^{i\theta}))\right|\right) \cdot \max_{1\leq k\leq n} \sup_{|\theta|\leq\pi} \left|\frac{dX_k(e^{i\theta})}{d\theta}\right| + \\
& \sup_{0<|\theta'-\theta''|\leq\pi} \sum_{k=1}^n \frac{\left|h_{x_k}(X(e^{i\theta'})) - h_{x_k}(X(e^{i\theta''}))\right|}{|\theta'-\theta''|^\alpha} \cdot \max_{1\leq k\leq n} \sup_{|\theta'|\leq\pi} \left|\frac{dX_k(e^{i\theta'})}{d\theta}\right| + \\
& \left(\sum_{k=1}^n \sup_{|\theta''|\leq\pi} \left|h_{x_k}(e^{i\theta''})\right|\right) \cdot \left(\max_{1\leq k\leq n} \sup_{0<|\theta'-\theta''|\leq\pi} \frac{\left|\frac{dX_k(e^{i\theta'})}{d\theta} - \frac{dX_k(e^{i\theta''})}{d\theta}\right|}{|\theta'-\theta''|^\alpha}\right),
\end{aligned}\right.$$

which yields the first inequality of (6.10) after using (6.2) for the second line of (6.11) and the trivial majoration $|\theta' - \theta''|^{1-\alpha} \leq \pi^{1-\alpha}$. The second and the third inequalities of (6.10) are established similarly, which completes the proof. □

The following direct consequence will be strongly used in Section 7 below.

Lemma 6.12. *Under the above assumptions, suppose that there exist constants* $c_1 > 0$, $K_2 > 0$ *with* $c_1 K_2 \leq \rho_1$ *such that for each* $c \in \mathbb{R}$ *with* $0 \leq c \leq c_1$*, there exists* $X_c \in \mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^n)$ *with* $||X_c||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c \cdot K_2$*. Then there exists a constant* $K_3 > 0$

such that the following three estimates hold:

$$(6.13)\qquad \begin{cases} ||h(X_c)||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c^{2+\alpha}\cdot K_3, \\ \sum_{k=1}^{n} ||h_{x_k}(X_c)||_{\mathcal{C}^{\alpha}(\partial\Delta)} \leq c^{1+\alpha}\cdot K_3, \\ \sum_{k_1,k_2=1}^{n} \left|\left|h_{x_{k_1}x_{k_2}}(X_c)\right|\right|_{\mathcal{C}^{\alpha}(\partial\Delta)} \leq c^{\alpha}\cdot K_3. \end{cases}$$

Proof. Applying Lemmas 6.4 and 6.9, we see that it suffices to choose

$$(6.14)\qquad K_3 := \max\left(K_1K_2^{2+\alpha}(3+\pi^{1-\alpha}),\ K_1K_2^{1+\alpha}(1+\pi^{1-\alpha}),\ 2K_1K_2^{\alpha}\right),$$

which completes the proof. □

Up to now, we have introduced three positive constants K_1, K_2, K_3. In Sections 7, 8 and 9 below, we shall introduce further positive constants K_4, K_5, K_6, K_7, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{16}, K_{17}, K_{18} and K_{19}, whose precise value will not be important.

§7. FAMILIES OF ANALYTIC DISCS HALF-ATTACHED TO MAXIMALLY REAL SUBMANIFOLDS

7.1. Preliminary. Let $E \subset \mathbb{C}^n$ be an arbitrary subset and let $A : \overline{\Delta} \to \mathbb{C}^n$ be a continuous mapping, holomorphic in Δ. If $\partial^+\Delta := \{\zeta \in \partial\Delta : \mathrm{Re}\,\zeta \geq 0\}$ denotes the *positive half-boundary* of Δ, we say that A is *half-attached* to E if $A(\partial^+\Delta) \subset E$. Such analytic discs which are glued in part to a geometric object were studied by S. Pinchuk in [P] to establish a boundary uniqueness principle about continuous functions on a maximally real submanifold of $\mathbb{C}^n$ which extend holomorphically to a wedge. Further works on the CR edge of the wedge theorem using discs partly attached to generic submanifolds were achieved by R. Ayrapetian [A] and by A. Tumanov [Tu2].

In this section, we shall construct local families of analytic discs $Z^1_{c,x,v}(\zeta) : \overline{\Delta} \to \mathbb{C}^n$, where $c \in \mathbb{R}^+$ is small, where $x \in \mathbb{R}^n$ is small and where $v \in \mathbb{R}^n$ is small, which are half-attached to a $\mathcal{C}^{2,\alpha}$-smooth maximally real submanifold M^1 of $\mathbb{C}^n$, which satisfy $Z^1_{c,0,v}(1) \equiv p_1 \in M^1$, such that the boundary point $Z^1_{c,x,v}(1)$ covers a neighborhood of p_1 in M^1 as x varies (c and v being fixed) and such that the tangent vector $\frac{\partial Z^1_{c,0,v}}{\partial\theta}(1)$ at the fixed point p_1 covers a cone in $T_{p_1}M^1$. These families will be used in Sections 8 and 9 below for the final steps in the proof of the main Proposition 5.12. With this choice, when x varies, v varies and ζ varies (but c is fixed), the set of points $Z^1_{c,x,v}(\zeta)$, covers a thin wedge of edge M^1 at p_1. Similar families of analytic discs were constructed in [BER] to reprove S. Pinchuk's boundary uniqueness theorem, with M^1 of class $\mathcal{C}^\infty$, using a method (implicit function theorem in Banach spaces) which in the case where M is of class $\mathcal{C}^{\kappa,\alpha}$ necessarily induces a loss of smoothness, yielding families of analytic discs which are only of class $\mathcal{C}^{\kappa-1,\alpha}$. Since we want our families to be of class at least $\mathcal{C}^2$ and since M is only of class $\mathcal{C}^{2,\alpha}$, we shall have to proceed differently.

To summarize symbolically the structure of the desired family:

$$(7.2)\qquad Z^1_{c,x,v}(\zeta) : \begin{cases} c = \text{small scaling factor}, \\ x = \text{translation parameter}, \\ v = \text{rotation parameter}, \\ \zeta = \text{unit disc variable}. \end{cases}$$

We shall begin our constructions in the "flat" case where the maximally real submanifold M^1 coincides with $\mathbb{R}^n$ and then perform a pertubation argument, using the scaling parameter c in an essential way.

7.3. A family of analytic discs sweeping $\mathbb{R}^n \subset \mathbb{C}^n$ with prescribed first order jets. We denote the coordinates over $\mathbb{C}^n$ by $z = x + iy = (x_1 + iy_1, \dots, x_n + iy_n)$. Let $c \in \mathbb{R}$ with $c \geq 0$ be a "scaling factor", let $n \geq 2$, let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and consider the algebraically parametrized family of analytic discs defined by

$$B_{c,x,v}(s+it) := (x_1 + cv_1(s+it), \dots, x_n + cv_n(s+it)), \tag{7.4}$$

where $s + it \in \mathbb{C}$ is the holomorphic variable. For $c \neq 0$, the map $B_{c,x,v}$ embeds the complex line $\mathbb{C}$ into $\mathbb{C}^n$ and sends $\mathbb{R}$ into $\mathbb{R}^n$ with *arbitrary first order jet at* 0: center point $B_{c,x,v}(0) = x$ and tangent direction $\partial B_{c,x,v}(s)/\partial s|_{s=0} = cv$.

To localize our family of analytic discs, we restrict the map (7.4) to the following specific set of values: $0 \leq c \leq c_0$ for some $c_0 > 0$; $|x| \leq c$; $|v| \leq 2$; and $|s+it| \leq 4$. To localize $\mathbb{R}^n$, we shall denote $M^0 := \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$, where $\rho_0 > 0$, and we notice that $B_{c,x,v}(\{|s+it| \leq 4\}) \subset M^0$ for all c, all x and all v provided that $c_0 \leq \rho_0/9$.

We then consider the mapping $(s+it) \longmapsto B_{c,x,v}(s+it)$ as a local (nonsmooth) analytic disc defined on the rectangle $\{s+it \in \mathbb{C} : |s| \leq 4, 0 \leq t \leq 4\}$ whose bottom boundary part $B_{c,x,v}([-4,4])$ is a small real segment contained in $\mathbb{R}^n$.

7.5. A useful conformal equivalence. Next, we have to get rid of the corners of the rectangle $\{s+it \in \mathbb{C} : |s| \leq 4, 0 \leq t \leq 4\}$. We proceed as follows. In the complex plane equipped with coordinates $s+it$, let $\mathcal{D}(i\sqrt{3}, 2)$ be the open disc of center $i\sqrt{3}$ and of radius 2. Let $\mu : (-2,2) \to [0,1]$ be an *even* $\mathcal{C}^\infty$-smooth function satisfying $\mu(s) = 0$ for $0 \leq s \leq 1$; $\mu(s) > 0$ and $d\mu(s)/ds > 0$ for $1 < s < 2$; and $\mu(s) = \sqrt{3} - \sqrt{4 - s^2}$ for $\sqrt{3} \leq s < 2$. The simply connected domain $C^+ \subset \{t > 0\}$ which is represented in FIGURE 1 just below may be formally defined as

$$\begin{cases} C^+ \cap \{t \geq \sqrt{3} - 1\} := \mathcal{D}(i\sqrt{3}, 2) \cap \{t \geq \sqrt{3} - 1\}, \\ C^+ \cap \{0 < t < \sqrt{3} - 1\} := \{s + it \in \mathbb{C} : t > \mu(s)\}. \end{cases} \tag{7.6}$$

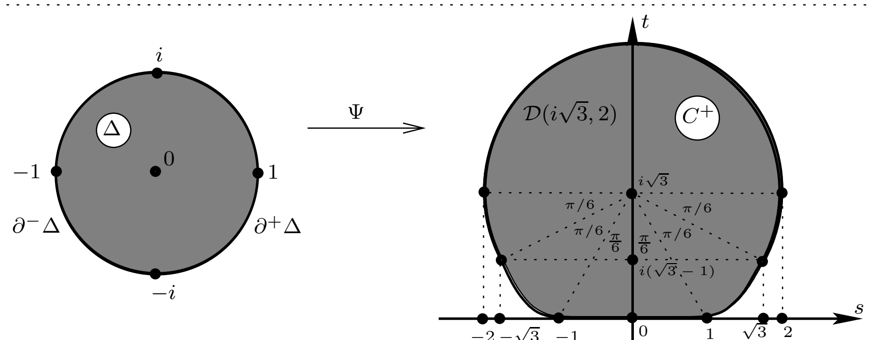

FIGURE 14: A USEFUL CONFORMAL MAPPING

Let $\Psi : \Delta \to C^+$ be a conformal equivalence (Riemann's theorem). Since the boundary ∂C^+ is $\mathcal{C}^\infty$-smooth, the mapping Ψ extends as a $\mathcal{C}^\infty$-smooth diffeomorphism $\partial\Delta \to \partial C^+$. Remind that $\partial^+\Delta := \{\zeta \in \mathbb{C} : |\zeta| = 1, \ \mathrm{Re}\,\zeta \geq 0\}$ is the positive half-boundary

of Δ. Then after a reparametrization of Δ, we can (and we shall) assume that $\Psi(\partial^+\Delta) = [-1,1]$, $\Psi(1) = 0$ and $\Psi(\pm i) = \pm 1$. It follows that $d\Psi\left(e^{i\theta}\right)/d\theta$ is a positive real number for all $e^{i\theta} \in \partial^+\Delta$. Although the precise shape of C^+ and the specific expression of Ψ will not be crucial in the sequel, it will be convenient to fix them once for all.

7.7. Flat families of half-attached analytic discs. Thanks to Ψ, we can define a family of small analytic discs which are half-attached to the "*flat*" maximally real manifold $M^0 \equiv \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$ as follows

$$Z^0_{c,x,v}(\zeta) := B_{c,x,v}\left(\Psi(\zeta)\right) = \left(x + cv\Psi(\zeta)\right). \tag{7.8}$$

We then have $Z^0_{c,x,v}(\partial^+\Delta) \subset M^0$ and $Z^0_{c,x,v}(1) = x$. Notice that every disc $Z_{c,x,v}\left(\overline{\Delta}\right)$ is contained in a single complex line. Starting with a maximally real submanifold of $\mathbb{C}^n$, as in Proposition 5.12, but dealing with the "flat" maximally real submanifold $M^0 \equiv \mathbb{R}^n$, we first construct a "flat model" of the desired family of analytic disc.

Lemma 7.9. *Let $M^0 = \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$ be the "flat" local maximally real submanifold defined above, let $p_0 \equiv 0 \in M^0$ denote the origin and let $v_0 \in T_{p_0}M^0$ be a tangent vector with $|v^0| = 1$. Then there exists a constant $\Lambda_0 > 0$ and there exists a $\mathcal{C}^\infty$-smooth family $A^0_{c,x,v}(\zeta)$ of analytic discs defined for $c \in \mathbb{R}$ with $0 \leq c \leq c_0$ for some $c_0 > 0$ satisfying $c_0 \leq \rho_0/9$, for $x \in \mathbb{R}^n$ with $|x| \leq c$ and for $v \in \mathbb{R}^n$ with $|v| \leq c$ which enjoy the following six properties*:

- $\mathbf{(1_0)}$ $A^0_{c,0,v}(1) = p_0 = 0$ *for all c and all v.*
- $\mathbf{(2_0)}$ $A^0_{c,x,v} : \overline{\Delta} \to \mathbb{C}^n$ *is an embedding and $\left|A^0_{c,x,v}(\zeta)\right| \leq c \cdot \Lambda_0$ for all c, all x, all v and all ζ.*
- $\mathbf{(3_0)}$ $A^0_{c,x,v}(\partial^+\Delta) \subset M^0$ *for all c, all x and all v.*
- $\mathbf{(4_0)}$ $\frac{\partial A^0_{c,0,0}}{\partial\theta}(1)$ *is a positive multiple of v_0 for all $c \neq 0$.*
- $\mathbf{(5_0)}$ *For all c, all v and all $e^{i\theta} \in \partial^+\Delta$, the mapping $x \longmapsto A^0_{c,x,v}\left(e^{i\theta}\right) \in M^0$ is of rank n.*
- $\mathbf{(6_0)}$ *For all $e^{i\theta} \in \partial^+\Delta$, all $c \neq 0$ and all x, the mapping $v \longmapsto \frac{\partial A^0_{c,x,v}}{\partial\theta}\left(e^{i\theta}\right)$ is of rank n at $v = 0$. Consequently, the positive half-lines $\mathbb{R}^+ \cdot \frac{\partial A_{c,0,v}}{\partial\theta}(1)$ describe an open infinite cone containing v_0 with vertex p_0 in $T_{p_0}M^0$ when v varies.*

Proof. Proceeding as in the proof of Lemma 5.37, we can find a new affine coordinate system centered at p_0 and stabilizing $\mathbb{R}^n$, which we shall still denote by $(z_1, \dots, z_n)$, in which the vector v_0 has coordinates $(0, \dots, 0, 1)$. In this coordinate system, we then construct the family $Z^0_{c,x,v}(\zeta)$ as in (7.8) above and we define the desired family simply as follows:

$$A^0_{c,x,v}(\zeta) := Z^0_{c,x,v_0+v}(\zeta), \tag{7.10}$$

where we restrict the variations of the parameter v to $|v| \leq c$. Notice that every disc $A^0_{c,x,v}\left(\overline{\Delta}\right)$ is contained in a single complex line. All the properties are then elementary consequences of the explicit expression (7.8) of $Z^0_{c,x,v}(\zeta)$.

Finally, we notice that it follows from properties $\mathbf{(5_0)}$ and $\mathbf{(6_0)}$ that the set of points $A^0_{c,x,v}(\zeta)$, where $c > 0$ is fixed, where x varies, where v varies and where ζ varies covers a local wedge of edge M^0 at p_0. The proof of Lemma 7.9 is complete. □

7.11. Curved families of half-attached analytic discs. Our main goal in this section is to obtain a statement similar to Lemma 7.9 after replacing the "flat" maximally real submanifold $M^0 \cong \mathbb{R}^n$ by a "curved" $\mathcal{C}^{2,\alpha}$-smooth maximally real submanifold M^1. We set up a formulation which will be appropriate for the achievement of the end of the proof of Proposition 5.12 in the next Sections 8 and 9. In particular, we shall have to shrink the family of half-disc $Z^1_{c,x,v}(\zeta)$ which we will construct as a perturbation of the family $Z^0_{c,x,v}(\zeta)$ in §7.50 below, and we shall construct discs of size $\leq c^2 \cdot \Lambda_1$ for some constant $\Lambda_1 > 0$, instead of requiring that their size is $\leq c \cdot \Lambda_1$, which would be the property analogous to **($\mathbf{2_0}$)**. Also, we shall loose the $\mathcal{C}^{2,\alpha-0}$-smoothness with respect to the scaling parameter c.

Lemma 7.12. *Let M^1 be $\mathcal{C}^{2,\alpha}$-smooth maximally real submanifold of $\mathbb{C}^n$, let $p_1 \in M^1$ and let $v_1 \in T_{p_1}M^1$ be a tangent vector with $|v_1| = 1$. Then there exists a positive constant $\Lambda_1 > 0$ and there exists $c_1 \in \mathbb{R}$ with $c_1 > 0$ such that for every $c \in \mathbb{R}$ with $0 < c \leq c_1$, there exists a family $A^1_{x,v:c}(\zeta)$ of analytic discs defined for $x \in \mathbb{R}^n$ with $|x| \leq c^2$ and for $v \in \mathbb{R}^n$ with $|v| \leq c$ which is $\mathcal{C}^{2,\alpha-0}$-smooth with respect to (x, v, ζ) and which enjoys the following six properties*:

- **($\mathbf{1_1}$)** $A^1_{0,v:c}(1) = p_1$ *for all* v.
- **($\mathbf{2_1}$)** $A^1_{x,v:c} : \overline{\Delta} \to \mathbb{C}^n$ *is an embedding and* $\left|A^1_{x,v:c}(\zeta)\right| \leq c^2 \cdot \Lambda_1$ *for all* x*, all* v *and all* ζ.
- **($\mathbf{3_1}$)** $A^1_{x,v:c}(\partial^+\Delta) \subset M^1$ *for all* x *and all* v.
- **($\mathbf{4_1}$)** $\frac{\partial A^1_{0,0:c}}{\partial \theta}(1)$ *is a positive multiple of* v_1 *for all* $c \neq 0$.
- **($\mathbf{5_1}$)** *The mapping* $x \longmapsto A^1_{x,0:c}(1) \in M^1$ *is of rank* n.
- **($\mathbf{6_1}$)** *The mapping* $v \longmapsto \frac{\partial A^1_{0,v:c}}{\partial \theta}\left(e^{i\theta}\right)$ *is of rank* n *at* $v = 0$*. Consequently, as* v *varies, the positive half-lines* $\mathbb{R}^+ \cdot \frac{\partial A^1_{0,v:c}}{\partial \theta}(1)$ *describe an open infinite cone containing* v_1 *with vertex* p_1 *in* $T_{p_1}M^1$ *and the set of points* $A^1_{x,v:c}(\zeta)$*, as* $|x| \leq c$*,* $|v| \leq c$ *and* $\zeta \in \Delta$ *vary, covers a wedge of edge* M^1 *at* (p_1, Jv_1).

In Figure 16 drawn in Section 8 after Lemma 8.3 below, we have drawn the property that the tangent direction $\frac{\partial A^1_{0,v:c}}{\partial \theta}(1)$ describes an open cone in $T_{p_1}M^1$ with vertex p_1. The remainder of Section 3 is devoted to complete the proof of Proposition 7.12.

7.13. Perturbed family of analytic discs half-attached to a maximally real submanifold. Thus, let $M^1 \subset \mathbb{R}^n$ be a locally defined maximally real $\mathcal{C}^{2,\alpha}$-smooth submanifold passing through the origin. We can assume that M^1 is represented by n Cartesian equations

$$y_1 = h_1(x_1, \ldots, x_n), \cdots\cdots, y_n = h_n(x_1, \ldots, x_n), \tag{7.14}$$

where $z_k = x_k + iy_k \in \mathbb{C}$, for $k = 1, \ldots, n$, where $|x| \leq \rho_1$ for some $\rho_1 > 0$, where $h = h(x)$ is of class $\mathcal{C}^{2,\alpha}$ in $\{|x| < 2\rho_1\}$, and where, importantly, $h_j(0) = h_{j,x_k}(0) = h_{j,x_{k_1}x_{k_2}}(0) = 0$, for all $j, k, k_1, k_2 = 1, \ldots, n$. As in (6.5), we set $K_1 := ||h||_{\mathcal{C}^{2,\alpha}(\{|x| \leq \rho_1\})}$. Also, we can assume that $v_1 = (0, \ldots, 0, 1)$.

Our goal is to show that we can produce a $\mathcal{C}^{2,\alpha-0}$-smooth (remind $\mathcal{C}^{2,\alpha-0} \equiv \bigcap_{\beta<\alpha} \mathcal{C}^{2,\beta}$) family of analytic discs $Z^1_{c,x,v}(\zeta)$ which is half-attached to M^1 and which is sufficiently close, in $\mathcal{C}^2$ norm, to the original family $Z^0_{c,x,v}(\zeta)$. After having constructed the family $Z^1_{c,x,v}(\zeta)$, we shall define the desired family $A^1_{x,v:c}(\zeta)$.

Let $d \in \mathbb{R}$ with $0 \leq d \leq 1$ and let the maximally real submanifold M^d (like "M *d*eformed") be defined precisely as the set of $z = x + iy \in \mathbb{C}^n$ with $|x| \leq \rho_1$ which satisfy the n Cartesian equations

$$y_1 = d \cdot h_1(x_1, \dots, x_n), \cdots\cdots, y_n = d \cdot h_n(x_1, \dots, x_n). \tag{7.15}$$

Notice that $M^0 \equiv \{x \in \mathbb{R}^n : \ |x| \leq \rho_1\}$ is essentially the same piece M^0 of $\mathbb{R}^n$ as in Lemma 7.9 (which contains the M^0 of Lemma 7.9 if we choose $\rho_0 \leq \rho_1$) and notice that $M^d\big|_{d=1} \equiv M^1$. Even better, we shall construct for each d with $0 \leq d \leq 1$ a one-parameter family of analytic dics $Z^d_{c,x,v}(\zeta)$ which is of class at least $\mathcal{C}^{2,\alpha-0}$ with respect to all variables and which is half-attached to M^d, by proceeding as follows.

First of all, the analytic disc $Z^d_{c,x,v}(\zeta) =: X^d_{c,x,v}(\zeta) + iY^d_{c,x,v}(\zeta)$ is half-attached to M^d if and only if

$$Y^d_{c,x,v}(\zeta) = d \cdot h\left(X^d_{c,x,v}(\zeta)\right), \quad \text{for } \zeta \in \partial^+\Delta. \tag{7.16}$$

Furthermore, $Y^d_{c,x,v}$ should be a harmonic conjugate of $X^d_{c,x,v}$. However, the condition (7.16) does not give any relation between $X^d_{c,x,v}$ and $Y^d_{c,x,v}$ on the negative part $\partial^-\Delta$ of the unit circle. To fix this point, we shall assign the following more complete equation

$$Y^d_{c,x,v}(\zeta) = d \cdot h\left(X^d_{c,x,v}(\zeta)\right) + Y^0_{c,x,v}(\zeta), \quad \text{for } \textit{all } \zeta \in \partial\Delta, \tag{7.17}$$

which coincides with (7.16) for $\zeta \in \partial^+\Delta$, since we have $Z^0_{c,x,v}(\partial^+\Delta) \subset \mathbb{R}^n$ by construction (*cf.* (7.8)).

As in [Tu2], [Tu3], [MP1], [MP3], we denote by T_1 the Hilbert transform (harmonic conjugate operator) on $\partial\Delta$ vanishing at 1, namely $(T_1 X)(1) = 0$, whence $T_1(T_1(X)) = -X + X(1)$. By Privalov's theorem, for every integer $\kappa \geq 0$ and every $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, its norm $|||T_1|||_{\kappa,\alpha}$ as an operator $\mathcal{C}^{\kappa,\alpha}(\partial\Delta, \mathbb{R}^n) \to \mathcal{C}^{\kappa,\alpha}(\partial\Delta, \mathbb{R}^n)$ is finite and explodes as α tends either to 0 or to 1. Also, we shall require that $X^d_{c,x,v}(1) = x$, whence $Y^d_{c,x,v}(1) = d \cdot h(x)$.

With this choice, the mapping $\zeta \mapsto Y^d_{c,x,v}(\zeta)$ should necessarily coincide with the harmonic conjugate $\zeta \mapsto \left[T_1 X^d_{c,x,v}\right](\zeta) + d \cdot h(x)$ (this property is already satisfied for $d = 0$) and we deduce that $X^d_{c,x,v}(\zeta)$ should satisfy the following Bishop type equation

$$X^d_{c,x,v}(\zeta) = -T_1\left[d \cdot h\left(X^d_{c,x,v}\right)\right](\zeta) + X^0_{c,x,v}(\zeta), \quad \text{for all } \zeta \in \partial\Delta. \tag{7.18}$$

Conversely, if $X^d_{c,x,v}$ is a solution of this functional equation, then setting $Y^d_{c,x,v}(\zeta) := T_1 X^d_{c,x,v}(\zeta) + d \cdot h(x)$, it is easy to see that the analytic disc $Z^d_{c,x,v}(\zeta) := X^d_{c,x,v}(\zeta) + iY^d_{c,x,v}(\zeta)$ is half-attached to M^d and more precisely, satisfies the equation (7.16).

Applying now Theorem 1.2 of [Tu3], we deduce that if the given positive number c_1 is sufficiently small, and if c satisfies $0 \leq c \leq c_1$, there exists a unique solution $X^d_{c,x,v}(\zeta)$ to (7.18) which is of class $\mathcal{C}^{2,\alpha}$ with respect to ζ and of class $\mathcal{C}^{2,\alpha-0}$ with respect to all variables (c, x, v, ζ) with $0 \leq c \leq c_1$, $|x| \leq c$, $|v| \leq 2$ and $\zeta \in \overline{\Delta}$. We shall now estimate the difference $||Z^d_{c,x,v} - Z^0_{c,x,v}||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}$ and prove that it is bounded by a constant times $c^{2+\alpha}$. In particular, if c_1 is sufficiently small, this will imply that $Z^d_{c,x,v}$ is nonconstant.

7.19. Size of the solution $X^d_{c,x,v}(\zeta)$ in $\mathcal{C}^{1,\alpha}$ norm. Following the beginning of the proof of Theorem 1.2 in [Tu3], we introduce the mapping

$$F : \ X(\zeta) \longmapsto X^0_{c,x,v}(\zeta) - T_1\left[d \cdot h(X)\right](\zeta) \tag{7.20}$$

from a neighborhood of 0 in $\mathcal{C}^{1,\alpha}(\partial\Delta,\mathbb{R}^n)$ to $\mathcal{C}^{1,\alpha}(\partial\Delta,\mathbb{R}^n)$, and then, as in [B], we introduce a Picard iteration processus by defining $X\{0\}^d_{c,x,v}(\zeta) := X^0_{c,x,v}(\zeta)$ and for every integer $\nu \geq 0$

$$X\{\nu+1\}^d_{c,x,v}(\zeta) := F\left(X\{\nu\}^d_{c,x,v}(\zeta)\right). \tag{7.21}$$

In a first moment, A. Tumanov proves in [Tu3] that the sequence $\left(X\{\nu\}^d_{c,x,v}(\zeta)\right)_{\nu\in\mathbb{N}}$ converges towards the unique solution $X^d_{c,x,v}(\zeta)$ of (7.18) in $\mathcal{C}^{1,\alpha}(\partial\Delta)$. Admitting this convergence result, we need to extract the supplementary information that $\left|\left|X^d_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c\cdot K_2$ for some positive constant K_2, which will play the role of the constant K_2 of Lemma 6.12.

To get this information, we observe that by construction (*cf.* (7.8)) there exists a constant $K_4 > 0$ such that

$$\left|\left|X^0_{c,x,v}\right|\right|_{\mathcal{C}^{2,\alpha}(\partial\Delta)} \leq c\cdot K_4. \tag{7.22}$$

Also, we set $K_5 := K_1(3+\pi^{1-\alpha})|||T_1|||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}$.

Lemma 7.23. *With these notations, if*

$$c_1 \leq \min\left(\frac{\rho_1}{2K_4},\ \left(\frac{1}{2^{2+\alpha}\,K_4^{1+\alpha}\,K_5}\right)^{\frac{1}{1+\alpha}}\right), \tag{7.24}$$

then the solution of (7.18) *satisfies* $\left|X^d_{c,x,v}\left(e^{i\theta}\right)\right| \leq \rho_1$ *for all* $e^{i\theta}\in\partial\Delta$ *and there exists a constant* $K_2>0$ *such that*

$$\left|\left|X^d_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c\cdot K_2. \tag{7.25}$$

In fact, it suffices to choose $K_2 := 2K_4$.

Proof. Indeed, using Lemmas 6.4 and 6.9, if $X\in\mathcal{C}^{1,\alpha}(\partial\Delta,\mathbb{R}^n)$ satisfies $\left|X\left(e^{i\theta}\right)\right|\leq\rho_1$ for all $e^{i\theta}\in\partial\Delta$ and $||X||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\leq c\cdot 2K_4$ for all $c\leq c_1$, where c_1 is as in (7.24), we may estimate (remind $0\leq d\leq 1$)

$$\left\{\begin{aligned} ||F(X)||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} &\leq \left|\left|X^0_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} + |||T_1|||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\cdot||h(X)||_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \\ &\leq c\cdot K_4 + |||T_1|||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\cdot K_1(c\cdot 2K_4)^{2+\alpha}(3+\pi^{1-\alpha}) \\ &= c\cdot\left(K_4 + c^{1+\alpha}2^{2+\alpha}K_4^{2+\alpha}K_5\right) \\ &\leq c\cdot(K_4 + c_1^{1+\alpha}2^{2+\alpha}K_4^{2+\alpha}K_5) \\ &\leq c\cdot 2K_4. \end{aligned}\right. \tag{7.26}$$

Notice that from the last inequality, it also follows that $\left|F\left(X\left(e^{i\theta}\right)\right)\right|\leq\rho_1$ for all $e^{i\theta}\in\partial\Delta$. Consequently, the processus of successive approximations (7.21) is well defined for each $\nu\in\mathbb{N}$ and from the inequality (7.26), we deduce that the limit $X^d_{c,x,v}$ satisfies the desired estimate $\left|\left|X^d_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\leq c\cdot 2K_4$, which completes the proof. □

Corollary 7.27. *Under the above assumptions, there exists a constant* $K_6>0$ *such that*

$$\left|\left|X^d_{c,x,v} - X^0_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c^{2+\alpha}\cdot K_6. \tag{7.28}$$

Proof. We estimate

$$\left\{\begin{aligned} \left|\left|X^d_{c,x,v} - X^0_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} &\leq |||T_1|||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\cdot\left|\left|h\left(X^d_{c,x,v}\right)\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \\ &\leq |||T_1|||_{\mathcal{C}^{1,\alpha}(\partial\Delta)}\cdot K_1(c\cdot 2K_4)^{2+\alpha}(3+\pi^{1-\alpha}) \\ &\leq c^{2+\alpha}\cdot K_5(2K_4)^{2+\alpha}. \end{aligned}\right. \tag{7.29}$$

so that it suffices to set $K_6 := K_5(2K_4)^{2+\alpha}$. □

7.30. Smallness of the deformation in $\mathcal{C}^2$ norm. As was already noticed (and admitted), the solution $X^d_{c,x,v}(\zeta)$ is in fact $\mathcal{C}^{2,\alpha}$-smooth with respect to ζ and $\mathcal{C}^{2,\alpha-0}$-smooth with respect to all variables (d,c,x,v,ζ). We can therefore differentiate twice Bishop's equation (7.18). First of all, if $X \in \mathcal{C}^{2,\alpha-0}(\partial\Delta,\mathbb{R}^n)$, we remind the commutation relation $\frac{\partial}{\partial\theta}(TX) = T\left(\frac{\partial X}{\partial\theta}\right)$, whence

$$\frac{\partial}{\partial\theta}\,(T_1X) = T\left(\frac{\partial X}{\partial\theta}\right), \tag{7.31}$$

since $T_1X = TX - TX(1)$. We may then compute the first order derivative of (7.18) with respect to θ:

(7.32)

$$\left\{ \frac{\partial}{\partial\theta}X^d_{c,x,v}\left(e^{i\theta}\right) - \frac{\partial}{\partial\theta}X^0_{c,x,v}\left(e^{i\theta}\right) = -T\left[d\cdot\sum_{l=1}^n \frac{\partial h}{\partial x_l}\left(X^d_{c,x,v}\right)\,\frac{\partial X^d_{l;c,x,v}}{\partial\theta}\right]\left(e^{i\theta}\right). \right.$$

and then its second order partial derivatives $\partial^2/\partial v_k\partial\theta$, for $k=1,\dots,n$, without writing the argument $e^{i\theta}$:

(7.33)

$$\left\{ \begin{aligned} \frac{\partial^2 X^d_{c,x,v}}{\partial v_k\partial\theta} - \frac{\partial^2 X^0_{c,x,v}}{\partial v_k\partial\theta} = -T\Bigg[& d\cdot\sum_{l_1,l_2=1}^n \frac{\partial^2 h}{\partial x_{l_1}\partial x_{l_2}}\left(X^d_{c,x,v}\right)\,\frac{\partial X^d_{l_1;c,x,v}}{\partial v_k}\,\frac{\partial X^d_{l_2;c,x,v}}{\partial\theta} + \\ & +d\cdot\sum_{l=1}^n \frac{\partial h_j}{\partial x_l}\left(X^d_{c,x,v}\right)\,\frac{\partial^2 X^d_{l;c,x,v}}{\partial v_k\partial\theta}\Bigg]. \end{aligned} \right.$$

Let now K_2 be as in (7.25) and let K_3 be as in Lemma 6.12, applied to $X^d_{c,x,v}(\zeta)$.

Lemma 7.34. *If in addition to the inequality* (7.24)*, the constant c_1 satisfies the inequality*

$$c_1 \leq \left(\frac{1}{2K_3|||T|||_{\mathcal{C}^\alpha(\partial\Delta)}}\right)^{\frac{1}{1+\alpha}}, \tag{7.35}$$

then there exists a positive constant $K_7>0$ such that for all d, all c, all x, all v, and for $k=1,\dots,n$, the following two estimates hold:

$$\left\{ \begin{aligned} \left|\left|\frac{\partial^2 X^d_{c,x,v}}{\partial v_k\partial\theta} - \frac{\partial^2 X^0_{c,x,v}}{\partial v_k\partial\theta}\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} &\leq c^{2+\alpha}\cdot K_7, \\ \left|\left|\frac{\partial^2 X^d_{c,x,v}}{\partial\theta^2} - \frac{\partial^2 X^0_{c,x,v}}{\partial\theta^2}\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} &\leq c^{2+\alpha}\cdot K_7. \end{aligned} \right. \tag{7.36}$$

Proof. We check only the first inequality, the proof of the second being totally similar. According to Lemma 1.6 in [Tu3], there exists a solution $\frac{\partial^2 X^d_{c,x,v}}{\partial v_k\partial\theta}$ to the linearized Bishop equation (7.33), hence it suffices to make an estimate.

Introducing for the second line of (7.33) a new simplified notation $\mathcal{R} := -T\left[d\cdot\sum_{l_1,l_2=1}^n \frac{\partial^2 h}{\partial x_{l_1}\partial x_{l_2}}\left(X^d_{c,x,v}\right)\,\frac{\partial X^d_{l_1;c,x,v}}{\partial v_k}\,\frac{\partial X^d_{l_2;c,x,v}}{\partial\theta}\right]$ and setting further obvious simplifying changes of notation, we can rewrite (7.33) more concisely as

$$\mathcal{X}^d - \mathcal{X}^0 = \mathcal{R} - T\left[d\cdot\mathcal{H}\mathcal{X}^d\right]. \tag{7.37}$$

Here, thanks to the inequality $\left|\left|X^d_{c,x,v}\right|\right|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} \leq c \cdot K_2$ already established in Lemma 7.23 and thanks to Lemma 6.12, we know that the vector $\mathcal{R} \in \mathcal{C}^\alpha(\partial\Delta, \mathbb{R}^n)$ and the matrix $\mathcal{H} \in \mathcal{C}^{1,\alpha}(\partial\Delta, \mathcal{M}_{n\times n}(\mathbb{R}))$ are small and more precisely, they satisfy the following two estimates

$$\left\{\begin{aligned} & ||\mathcal{R}||_{\mathcal{C}^\alpha(\partial\Delta)} \leq c^{2+\alpha} \cdot |||T|||_{\mathcal{C}^\alpha(\partial\Delta)} K_3 (K_2)^2 \\ & ||\mathcal{H}||_{\mathcal{C}^\alpha(\partial\Delta)} \leq c^{1+\alpha} \cdot K_3. \end{aligned}\right. \tag{7.38}$$

We can rewrite (7.37) under the form

$$\mathcal{X}^d - \mathcal{X}^0 = \mathcal{S} - T\left[d \cdot \mathcal{H}(\mathcal{X}^d - \mathcal{X}^0)\right], \tag{7.39}$$

with $\mathcal{S} := \mathcal{R} - T\left[d \cdot \mathcal{H}\mathcal{X}^0\right]$. Using the inequality $\left|\left|\mathcal{X}^0\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} \leq c \cdot K_4$ which is a direct consequence of (7.22) and taking the previous estimates (7.38) into account, we deduce the inequality

$$||\mathcal{S}||_{\mathcal{C}^\alpha(\partial\Delta)} \leq c^{2+\alpha} \cdot |||T|||_{\mathcal{C}^\alpha(\partial\Delta)} \left[K_3(K_2)^2 + K_3K_4\right]. \tag{7.40}$$

Taking the $\mathcal{C}^\alpha(\partial\Delta)$ norm of both sides of (7.40), we deduce the estimate

$$\left\{\begin{aligned} \left|\left|\mathcal{X}^d - \mathcal{X}^0\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} & \leq c^{2+\alpha} \cdot \frac{|||T|||_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right]}{1 - c^{1+\alpha} \cdot |||T|||_{\mathcal{C}^\alpha(\partial\Delta)} K_3} \\ & \leq c^{2+\alpha} \cdot 2|||T|||_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right] \end{aligned}\right. \tag{7.41}$$

where we use the assumption (7.35) on c_1 to obtain the second second inequality. It suffices to set $K_7 := 2|||T|||_{\mathcal{C}^\alpha(\partial\Delta)}\left[K_3(K_2)^2 + K_3K_4\right]$, which completes the proof. □

7.42. Adjustment of the tangent vector. Let $v_1 \in T_{p_1}M^1$ with $|v_1| = 1$, as in Lemma 7.12. Coming back to the first family $Z^0_{c,x,v}(\zeta)$ defined by (7.8), we observe that

$$\left\{\begin{aligned} & \frac{\partial Z^0_{j;c,0,v_1}}{\partial x_k}(1) = \delta^j_k, \qquad j,k = 1,\dots,n, \\ & \frac{\partial^2 Z^0_{j;c,0,v_1}}{\partial v_k \partial\theta}(1) = c\,\frac{\partial\Psi}{\partial\theta}\left(e^{i\theta}\right)\,\delta^j_k, \qquad j,k = 1,\dots,n. \end{aligned}\right. \tag{7.43}$$

From now on, we shall set $d = 1$ and we shall only consider the family $Z^1_{c,x,v}(\zeta)$. Thanks to the estimates (7.28) and (7.36), we deduce that if c_1 is sufficiently small, then for all c with $0 < c \leq c_1$, the two Jacobian matrices

$$\left(\frac{\partial Z^1_{j;c,0,v_1}}{\partial x_k}(1)\right)_{1\leq j,k\leq n} \qquad \text{and} \qquad \left(\frac{\partial^2 Z^1_{j;c,0,v_1}}{\partial v_k\partial\theta}(1)\right)_{1\leq j,k\leq n} \tag{7.44}$$

are invertible. It would follow that if we would set $A^1_{x,v:c}(\zeta) := Z^1_{c,x,v_1+v}(\zeta)$, similarly as in (7.10), then the disc $A^1_{x,v:c}(\zeta)$ would satisfy the two rank properties $\mathbf{(5_1)}$ and $\mathbf{(6_1)}$ of Lemma 7.12. However, the tangency condition $\mathbf{(4_1)}$ would certainly not be satisfied, because as d varies from 0 to 1, the disc $Z^d_{c,x,v}(\zeta)$ undergo a nontrivial deformation.

Consequently, for every c with $0 < c \leq c_1$, we have to adjust the "cone parameter" v in order to maintain the tangency condition.

Lemma 7.45. *For every c with $0 < c \leq c_1$, there exists a vector $v(c) \in \mathbb{R}^n$ such that*

$$\frac{\partial Z^1_{c,0,v_1+v(c)}}{\partial\theta}(1) = \frac{\partial Z^0_{c,0,v_1}}{\partial\theta}(1) = c \cdot \frac{\partial\Psi}{\partial\theta}(1)\cdot v_1. \tag{7.46}$$

Furthermore, there exists a constant $K_8 > 0$ *such that* $|v(c)| \leq c^{1+\alpha} \cdot K_8$.

Proof. Unfortunately, we cannot apply the implicit function theorem, because the mapping $Z^1_{c,x,v}$ is identically zero when $c = 0$, so we have to proceed differently. First, we set

$$C_1 := \frac{\partial \Psi}{\partial \theta}(1), \quad \text{and} \quad C_2 := ||\Psi||_{\mathcal{C}^2(\overline{\Delta})}. \tag{7.47}$$

The constant C_2 will be used only in Section 8 below. Choose $K_8 \geq \frac{2K_6}{C_1}$. According to the explicit expression (7.8), the set of points

$$\left\{ \frac{\partial X^0_{c,0,v_1+v}}{\partial \theta}(1) \in \mathbb{R}^n : \ |v| \leq c^{1+\alpha} \cdot K_8 \right\} \tag{7.48}$$

covers a cube in $\mathbb{R}^n$ centered at the point $\frac{\partial X^0_{c,0,v_1}}{\partial \theta}(1)$ of radius $c^{2+\alpha} \cdot C_1 K_8$. Thanks to the estimate (7.28), we deduce that the (deformed) set of points

$$\left\{ \frac{\partial X^1_{c,0,v_1+v}}{\partial \theta}(1) \in \mathbb{R}^n : \ |v| \leq c^{1+\alpha} \cdot K_8 \right\} \tag{7.49}$$

covers a cube in $\mathbb{R}^n$ centered at the same point $\frac{\partial X^0_{c,0,v_1}}{\partial \theta}(1)$, but of radius

$$c^{2+\alpha} \cdot C_1 K_8 - c^{2+\alpha} \cdot K_6 \geq c^{2+\alpha} \cdot K_6. \tag{7.50}$$

Consequently, there exists at least one $v(c) \in \mathbb{R}^n$ with $|v(c)| \leq c^{1+\alpha} \cdot K_8$ such that (7.46) holds, which completes the proof. □

7.51. Construction of the family $A^1_{x,v:c}(\zeta)$**.** We can now complete the proof of the main Lemma 7.12 of the present section. First of all, with $\Psi(\zeta)$ as in §7.5 ans in FIGURE 14, we consider the composed conformal mapping

$$\zeta \longmapsto c\Psi(\zeta) \longmapsto \frac{i - c\Psi(\zeta)}{i + c\Psi(\zeta)} =: \Phi_c(\zeta). \tag{7.52}$$

The image $\Phi_c(\zeta)$ of the unit disc is a small domain contained in Δ and concentrated near 1. More precisely, assuming that c satifies $0 < c \leq c_1$ with $c_1 << 1$ as in the previous paragraphs, and taking account of the definition of $\Psi(\zeta)$, it can be checked easily that $\Phi_c(1) = 1$, that $\Phi_c(\partial^+\Delta)$ is contained in $\{e^{i\theta} \in \partial^+\Delta : \ |\theta| < 10c\}$, and that

(7.53)

$$\Phi_c\left(\overline{\Delta} \backslash \partial^+ \Delta\right) \subset \{\zeta \in \Delta : \ |\zeta - 1| < 8c\} \subset \{\rho e^{i\theta} \in \Delta : \ |\theta| < 10c, \ 1 - 10c < \rho < 1\}.$$

the second inclusion being trivial. Here is an illustration:

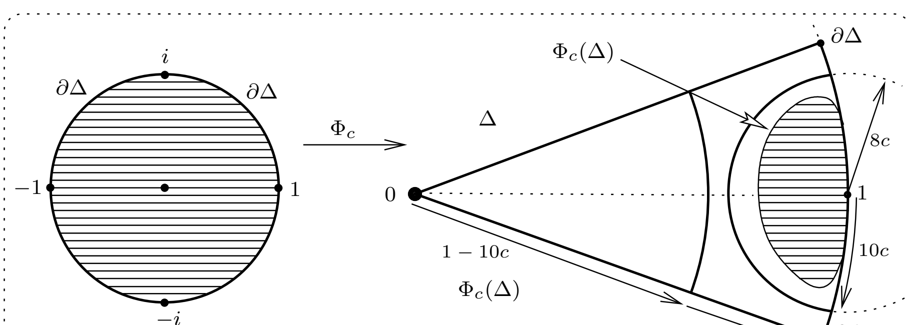

FIGURE 15: A SUB-HALF-DISC

We can now define the final desired family of analytic discs, writing the parameter c after a semi-colon, since we lose the $\mathcal{C}^{2,\alpha-0}$-smoothness with respect to c after the application of Lemma 7.45:

$$A^1_{x,v:c}(\zeta) := Z^1_{c,x,v_1+v(c)+v}\left(\Phi_c(\zeta)\right). \tag{7.54}$$

We restrict the variation of the parameters x to $|x| \leq c^2$ and v to $|v| \leq c$. Property $\mathbf{(4_1)}$ holds immediately, thanks to the choice of $v(c)$. Properties $\mathbf{(1_1)}$, $\mathbf{(3_1)}$, $\mathbf{(5_1)}$ and $\mathbf{(6_1)}$ as well as the embedding property in $\mathbf{(2_1)}$ are direct consequences of the similar properties (7.44) satisfied by $Z^1_{c,x,v_1+v(c)+v}(\zeta)$, using the chain rule and the nonvanishing of the partial derivative $\frac{\partial \Phi_c}{\partial \theta}(1)$. The size estimate in $\mathbf{(2_1)}$ follows from (7.25), from (7.28), from the restriction of the domains of variation of x and of v and from (7.53). This completes the proof of Lemma 7.12. □

§8. Geometric properties of families of half-attached analytic discs

8.1. Preliminary. By Lemma 7.12, for every c with $0 < c \leq c_1$, the family of half-attached analytic discs $A^1_{x,v:c}(\zeta)$ covers a local wedge of edge M^1 at p_1. However, not only we want the family $A^1_{x,v:c}$ to cover a local wedge of edge M^1 at p_1, but we certainly want to remove the point p_1 by means of the continuity principle, under the assumptions of the main Proposition 5.12, a final task which will be achieved in Section 9 below. Consequently, in each one of the three geometric situations $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ and **(II)** which we have normalized in Lemma 5.37 above, we shall firstly deduce from the tangency condition $\mathbf{(4_1)}$ of Lemma 7.12 that the blunt half-boundary $A^1_{0,0:c}(\partial^+\Delta\backslash\{1\})$ is contained in the open side $(H^1)^+$ (this is why we have normalized in Lemma 5.37 the second order terms of the supporting hypersurface H^1 in order that $(H^1)^+$ is strictly concave; the reason why we require that $A^1_{0,0:c}(\partial^+\Delta\backslash\{1\})$ is contained in $(H^1)^+$ will be clear in Section 9 below). Secondly, we shall show that for all x with $|x| \leq c^2$, the disc interior $A_{x,0:c}(\Delta)$ is contained in the local half-wedge $\mathcal{HW}_1^+$ in the cases $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ and is contained in the wedge $\mathcal{W}_2$ in case **(II)**.

8.2. Geometric disposition of the discs with respect to H^1 and to $\mathcal{HW}_1^+$ or to $\mathcal{W}_2$. We remember that the positive c_1 of Lemmas 7.12, 7.23 and 7.34 was shrunk explicitely, in terms of the constants $K_1, K_2, K_3, \dots$. In this section, we shall again shrink c_1 several times, but without mentioning all the similar explicit inequalities which will appear. The precise statement of the main lemma of this section, which is a continuation of Lemma 7.12, is as follows; whereas we can essentially gather the three cases in the formal statement of the lemma, it is necessary to treat them separately in the proof, because the normalizations of Lemma 5.37 differ.

Lemma 8.3. *Let M, let M^1, let p_1, let H^1, let v_1, let $(H^1)^+$, let $\mathcal{HW}_1^+$ or let $\mathcal{HW}_2$ and let a coordinate system $z = (z_1, \dots, z_n)$ vanishing at p_1 be as in Case $\mathbf{(I_1)}$, as in Case $\mathbf{(I_2)}$ or as in Case **(II)** of Lemma 5.37. Choose as a local one-dimensional submanifold $T^1 \subset M^1$ transversal to H^1 in M^1 and passing through p_1 the submanifold $T_1 := \{(x_1, 0, \dots, 0) + ih(x_1, 0, \dots, 0))\}$ in Case $\mathbf{(I_1)}$ and the submanifold $T_1 := \{(0, \dots, 0, x_n) + ih(0, \dots, 0, x_n))\}$ in Cases $\mathbf{(I_2)}$ and **(II)**. For every c with $0 < c \leq c_1$, let $A^1_{x,v:c}(\zeta)$ be the family of analytic discs satisfying properties $\mathbf{(1_1)}$, $\mathbf{(2_1)}$, $\mathbf{(3_1)}$, $\mathbf{(4_1)}$, $\mathbf{(5_1)}$ and $\mathbf{(6_1)}$ of Lemma 7.12. Shrinking c_1 if necessary, then for every c with $0 < c \leq c_1$, the following three further properties hold*

$\mathbf{(7_1)}$ $A^1_{0,0:c}(\partial^+\Delta\backslash\{1\}) \subset (H^1)^+$.

(8_1) *$A^1_{x,0:c}(\partial^+\Delta)$ is contained in $(H^1)^+$ for all x such that the point $A^1_{x,0:c}(1)$ belongs to $T^1 \cap (H^1)^+$.*

(9_1) *$A^1_{x,v:c}(\overline{\Delta}\backslash\partial^+\Delta)$ is contained in the half-wedge $\mathcal{HW}_1^+$ or in the wedge $\mathcal{W}_2$ for all x and all v.*

Proof. For the three new properties **(7_1)**, **(8_1)**, and **(9_1)**, we study thoroughly only Case **($\mathbf{I_1}$)**, because the other two cases can be treated in a totally similar way. FIGURE 16 just below illustrates properties **(7_1)** and **(8_1)** and also properties **(1_1)**, **(5_1)** and **(6_1)** of Lemma 7.12.

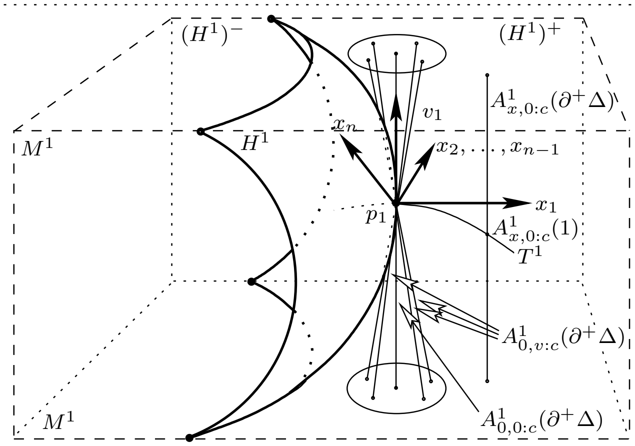

FIGURE 16: VIEW OF THE FAMILY OF HALF BOUNDARIES IN M^1

Intuitively, the reason why property **(7_1)** holds true is clear: the open set $(H^1)^+$ is strictly concave and the small segment $A^1_{0,0:c}(\partial^+\Delta)$ is tangent to H^1 at p_1; also, the reason why property **(8_1)** holds true is equally clear: when x varies, the small segments $A^1_{x,0:c}(\partial^+\Delta)$ are essentially translated (inside M^1) from p_1 by the vector $x \in \mathbb{R}^n$; and finally, the reason why property **(8_1)** holds true has a simple geometric interpretation: if the scaling parameter c_1 is small enough, the small analytic disc $A^1_{x,v:c}(\overline{\Delta})$ is essentially a slightly deformed small part of the straight complex line $\mathbb{C}\cdot(v_1 + Jv_1)$, where the half-wedge $\mathcal{HW}_1^+$ or the wedge $\mathcal{W}_2$ is directed by the vector Jv_1 according to Lemma 5.37. The next paragraphs are devoted to some elementary estimates which will establish these properties rigorously.

Firstly, let us prove property **(7_1)** in Case **($\mathbf{I_1}$)**. According to Lemma 5.37, the vector v_1 is given by $(0, 1, \dots, 1)$ and the side $(H^1)^+ \subset M^1$ is defined by $x_1 > g(x') = -x_2^2 - \cdots - x_n^2 + \widehat{g}(x')$, where the $\mathcal{C}^{2,\alpha}$-smooth function $\widehat{g}(x')$ vanishes to second order at the origin, thanks to the normalization conditions (5.40). By Lemma 6.4, the remainder $\widehat{g}(x')$ then satisfies an inequality of the form $|\widehat{g}(x')| \leq K_9 \cdot |x'|^{2+\alpha} \leq K_9 \cdot \left(\sum_{j=2}^n x_j^2\right)^{\frac{\alpha+2}{2}}$, for some constant $K_9 > 0$. Since the strictly concave open subset $(\widetilde{H}^1)^+$ of M^1 with $\mathcal{C}^{2,\alpha}$-smooth boundary defined by the inequality $x_1 > -x_1^2 - \cdots - x_n^2 + K_9 \cdot \left(\sum_{j=2}^n x_j^2\right)^{\frac{2+\alpha}{2}}$ is contained in $(H^1)^+$, it suffices to prove property **(7_1)** with $(H^1)^+$ replaced by $(\widetilde{H}^1)^+$.

By construction, the disc boundary $A_{0,0:c}(\partial\Delta)$ is tangent at p_1 to H^1, hence also to $\widetilde{H}^1$. Intuitively, it is clear that the blunt disc half-boundary $A_{0,0:c}(\partial^+\Delta\backslash\{1\})$ should then be contained in the strictly concave open subset $(\widetilde{H}^1)^+$, *see* FIGURE 16 above.

To proceed rigorously, we shall come back to the definition (7.53) which yields $A^1_{0,0:c}(\zeta) \equiv Z^1_{c,0,v_1+v(c)}(\Phi_c(\zeta))$, with the tangency condition (7.46) satisfied. First of all, denoting the n components of $v(c)$ by $(v_1(c),\dots,v_n(c))$, we may compute the second order derivatives of the similar discs attached to M^0:

$$
\tag{8.4}
\begin{cases}
\dfrac{\partial^2 Z^0_{1;c,0,v_1+v(c)}}{\partial\theta^2}(1) = c\cdot\dfrac{\partial^2\Psi}{\partial\theta^2}\left(e^{i\theta}\right)\cdot v_1(c),\\
\dfrac{\partial^2 Z^0_{j;c,0,v_1+v(c)}}{\partial\theta^2}(1) = c\cdot\dfrac{\partial^2\Psi}{\partial\theta^2}\left(e^{i\theta}\right)\cdot(1+v_j(c)), \qquad j=2,\dots,n.
\end{cases}
$$

Using the definition (7.47), the inequality $|v(c)| \leq c^{1+\alpha}\cdot K_8$ and the second estimate (7.36), we deduce that

$$
\tag{8.5}
\begin{cases}
\left|\dfrac{\partial^2 Z^1_{1;c,0,v_1+v(c)}}{\partial\theta^2}(1)\right| \leq c^{2+\alpha}\cdot K_7 + c^{2+\alpha}\cdot C_2K_8 =: c^{2+\alpha}\cdot 2K_{10}\\
\left|\dfrac{\partial^2 Z^1_{j;c,0,v_1+v(c)}}{\partial\theta^2}(1)\right| \leq c\cdot 2C_2, \qquad j=2,\dots,n.
\end{cases}
$$

Applying then Taylor's integral formula $F(\theta)=F(0)+\theta\cdot F'(0)+\int_0^\theta(\theta-\theta')\cdot\partial_\theta\partial_\theta F(\theta')\cdot d\theta'$ to $F(\theta) := X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right)$ and afterwards to $F(\theta) := X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right)$ for $j=2,\dots,n$, taking account of the tangency conditions

$$
\tag{8.6}
\frac{\partial X^1_{1;c,0,v_1+v(c)}}{\partial\theta}(1) = 0, \qquad\qquad \frac{\partial X^1_{j;c,0,v_1+v(c)}}{\partial\theta}(1) = c\cdot C_1, \qquad j=2,\dots,n,
$$

(a simple rephrasing of (7.46)) and using the inequalities (8.5), we deduce that

$$
\tag{8.7}
\begin{cases}
\left|X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right| \leq \theta^2\cdot c^{2+\alpha}\cdot K_{10},\\
\left|X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right) - \theta\cdot c\cdot C_1\right| \leq \theta^2\cdot c\cdot C_2, \qquad j=2,\dots,n.
\end{cases}
$$

Recall that

$$
\tag{8.8}
x_1 > \widetilde{g}(x') := -x_2^2 - \cdots - x_n^2 + K_9\left(\sum_{j=2}^n x_j^2\right)^{\frac{2+\alpha}{2}}
$$

denotes the equation of $(\widetilde{H}^1)^+$. We now claim that if c_1 is sufficiently small, then for every θ with $0<|\theta|<10c$, we have

$$
\tag{8.9}
X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right) > \widetilde{g}\left(X^1_{2;c,0,v_1+v(c)}\left(e^{i\theta}\right),\dots\dots, X^1_{n;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right).
$$

Since $\Phi_c(\partial^+\Delta)$ is contained in $\{e^{i\theta}\in\partial^+\Delta : |\theta|<10c\}$, this will imply the desired inclusion for proving $\mathbf{(7_1)}$:

$$
\tag{8.10}
\begin{cases}
A^1_{x,v:c}(\partial^+\Delta\backslash\{1\}) = Z^1_{c,0,v_1+v(c)}\left(\Phi_c(\partial^+\Delta\backslash\{1\})\right)\subset\\
\qquad\subset Z^1_{c,0,v_1+v(c)}\left(\{e^{i\theta}\in\partial^+\Delta:\ 0<|\theta|\leq 10c\}\right)\subset(\widetilde{H}^1)^+.
\end{cases}
$$

To prove the claim, we notice a minoration of the left hand side of (8.9), using (8.7)

$$X^1_{1;c,0,v_1+v(c)}\left(e^{i\theta}\right) \geq -\theta^2 \cdot c^{2+\alpha} \cdot K_{10}. \tag{8.11}$$

On the other hand, using two inequalities which are direct consequences of the second line of (8.7), provided that $10c_1 \cdot C_2 \leq \frac{C_1}{2}$:

$$\begin{cases} \left|X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right| \leq |\theta| \cdot c \cdot (C_1 + |\theta| \cdot C_2) \leq |\theta| \cdot c \cdot \dfrac{3C_1}{2}, \\ \left[X^1_{j;c,0,v_1+v(c)}\right]^2 \geq \theta^2 \cdot c^2 \cdot (C_1 - |\theta| \cdot C_2)^2 \geq \theta^2 \cdot c^2 \cdot \dfrac{C_1^2}{4}, \end{cases} \tag{8.12}$$

for $j = 2, \ldots, n$, we deduce the following majoration of the right hand side of (8.9)

$$\begin{cases} \widetilde{g}\ \left(X^1_{2;c,0,v_1+v(c)}\left(e^{i\theta}\right), \ldots\ldots, X^1_{n;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right) = \\ = -\displaystyle\sum_{j=2}^{n} \left[X^1_{j;c,0,v_1+v(c)}\right]^2 + K_9 \left(\sum_{j=2}^{n} \left[X^1_{j;c,0,v_1+v(c)}\left(e^{i\theta}\right)\right]^2\right)^{\frac{2+\alpha}{2}} \\ \leq -\theta^2 \cdot c^2 \cdot \dfrac{C_1^2}{4}(n-1) + |\theta|^{2+\alpha} \cdot c^{2+\alpha} \cdot \left(\dfrac{(n-1)9C_1^2}{4}\right)^{\frac{2+\alpha}{2}} K_9 \\ \leq -\theta^2 \cdot c^2 \left(\dfrac{C_1^2}{4}(n-1) - c^\alpha \cdot \left(\dfrac{(n-1)9C_1^2}{4}\right)^{\frac{2+\alpha}{2}} K_9\right). \end{cases} \tag{8.13}$$

Thanks to the minoration (8.11) and to the majoration (8.13), in order that the inequality (8.9) holds for all θ with $0 < |\theta| \leq 10c$, it suffices that the right hand side of (8.11) be greater than the last line of (8.13). By writing this strict inequality and clearing the factor $\theta^2 \cdot c^2$, we see that it suffices that

$$-K_{10} \cdot c^\alpha > -\left(\frac{C_1^2}{4}(n-1) - c^\alpha \cdot \left(\frac{(n-1)9C_1^2}{4}\right)^{\frac{2+\alpha}{2}} K_9\right), \tag{8.14}$$

or equivalently

$$c_1 < \left(\frac{\frac{C_1^2}{4}(n-1)}{K_{10} + \left(\frac{(n-1)9C_1^2}{4}\right)^{\frac{2+\alpha}{2}} K_9}\right)^{\frac{1}{\alpha}}. \tag{8.15}$$

This completes the proof of property $\mathbf{(7_1)}$.

Secondly, let us prove property $\mathbf{(8_1)}$ in Case $\mathbf{(I_1)}$, proceeding similarly. As above, we come back to the definition $A^1_{x,0:c}(\zeta) := Z^1_{c,x,v_1+v(c)}\left(\Phi_c(\zeta)\right)$ and we remind that $A^1_{x,0:c}(1) = Z^1_{c,x,v_1+v(c)}(1) = x + ih(x)$, which follows by putting $d = 1$ and $\zeta = 1$ in (7.18). Thanks to the inclusion $\Phi_c(\partial^+\Delta) \subset \{e^{i\theta} \in \partial^+\Delta : \ |\theta| < 10c\}$, it suffices to prove that the segment $Z_{c,x,v_1+v(c)}\left(\left\{e^{i\theta} : \ |\theta| < 10c\right\}\right)$ is contained in the open side $(\widetilde{H}^1)^+ \subset (H^1)^+$ defined by the inequation (8.8), if the point $x + ih(x)$ belongs to the transversal half-submanifold $T^1 \cap (H^1)^+$, namely if $x = (x_1, 0, \ldots, 0)$ with $x_1 > 0$. In the sequel, we shall denote the disc $Z^1_{c,x,v_1+v(c)}(\zeta)$ by $Z^1_{c,x_1,x',v_1+v(c)}(\zeta)$, emphasizing the decomposition $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, and we shall also use the convenient

notation
(8.16)
$$Z'^{1}_{c,x_1,x',v_1+v(c)}\left(\rho e^{i\theta}\right) := \left(Z^1_{2;c,x_1,x',v_1+v(c)}\left(\rho e^{i\theta}\right),\ldots\ldots, Z^1_{n;c,x_1,x',v_1+v(c)}\left(\rho e^{i\theta}\right)\right).$$

So, we have to show that for all c with $0 < c \leq c_1$, for all x_1 with $0 < x_1 \leq c^2$ and for all θ with $|\theta| < 10c$, then the following strict inequality holds true

$$X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) > \widetilde{g}\left(X^{1'}_{c;x_1,0,v_1+v(c)}\left(e^{i\theta}\right)\right), \tag{8.17}$$

First of all, coming back to the family of discs attached to M^0, we see by differentiating (7.8) twice with respect to x_1 that $\frac{\partial^2 Z^0_{c,x_1,0,v_1+v(c)}}{\partial x_1^2}(\zeta) \equiv 0$. Next, by differentiating twice Bishop's equation (7.18) with respect to x_1 and by reasoning as in Lemma 7.34, we deduce the estimate

$$\left|\left|\frac{\partial^2 Z^1_{c,x_1,0,v_1+v(c)}}{\partial x_1^2}\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} \leq c^{2+\alpha}\cdot K_7, \tag{8.18}$$

say, with the same constant $K_7 > 0$ as in Lemma 7.34, after enlarging it if necessary. Applying then Taylor's integral formula $F(x_1) = F(0) + x_1 \cdot \partial_{x_1} F(0) + \int_0^{x_1}(x_1 - \widetilde{x}_1)\cdot \partial_{x_1}\partial_{x_1} F(\widetilde{x}_1)\cdot d\widetilde{x}_1$ to the function $F(x_1) := X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right)$, we deduce the minoration
(8.19)
$$X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) \geq X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1\cdot\frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right) - x_1^2\cdot c^{2+\alpha}\cdot\frac{K_7}{2}.$$

On the other hand, by differentiating Bishop's equation (7.18) with respect to x_1 at $x = 0$, the derivative $\partial_{x_1} x$ yields the vector $(1, 0, \ldots, 0)$ and we obtain
(8.20)
$$\left\{\begin{aligned} \frac{\partial X_{c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right) = &-T_1\left[\sum_{l=1}^n \frac{\partial h}{\partial x_l}\left(X^1_{c,0,0,v_1+v(c)}(\cdot)\right)\,\frac{\partial X^1_{l;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot)\right](e^{i\theta}) + \\ &+ (1, 0, \ldots, 0). \end{aligned}\right.$$

Using then the second inequality (6.13) and the estimate (7.25), we deduce from (8.20)
(8.21)
$$\left\{\begin{aligned} \left|\left|\frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot) - 1\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} &\leq c^{2+\alpha}\cdot |||T_1|||_{\mathcal{C}^\alpha(\partial\Delta)} K_2 K_3, \\ \left|\left|\frac{\partial X^1_{j;c,0,0,v_1+v(c)}}{\partial x_1}(\cdot)\right|\right|_{\mathcal{C}^\alpha(\partial\Delta)} &\leq c^{2+\alpha}\cdot |||T_1|||_{\mathcal{C}^\alpha(\partial\Delta)} K_2 K_3, \qquad j = 2, \ldots, n. \end{aligned}\right.$$

Thanks to the first line of (8.21), we can refine the minoration (8.19) by replacing the first order partial derivative $\frac{\partial X^1_{1;c,0,0,v_1+v(c)}}{\partial x_1}\left(e^{i\theta}\right)$ in the right hand side of (8.19) by the constant 1, modulo an error term and also, we can use the trivial minoration $-x_1^2 \geq -x_1$, which yields a new, more interesting minoration of the form

$$X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) \geq X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1 - x_1\cdot c^{2+\alpha}\cdot K_{11}, \tag{8.22}$$

for some constant $K_{11} > 0$. On the other hand, using the inequalities $|\partial_{x_j}\widetilde{g}(x')| \leq |x'| + K_9\cdot|x'|^{1+\alpha}\cdot\left(1+\frac{\alpha}{2}\right)(n-1)^{\frac{\alpha}{2}}$ for $j = 2, \ldots, n$, using the estimate (7.25) and

using (6.2), we deduce an inequality of the form

$$\widetilde{g}\left(X'^{1}_{c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right)\right) \leq \widetilde{g}\left(X'^{1}_{c,0,0,v_1+v(c)}\left(e^{i\theta}\right)\right) + x_1 \cdot c \cdot K_{12}, \tag{8.23}$$

for some constant $K_{12} > 0$. Finally, putting together the two inequalities (8.22) and (8.23), and using the following inequality, which is an immediate consequence of the strict inequality (8.9):

$$X_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) \geq \widetilde{g}\left(X'^{1}_{c,0,0,v_1+v(c)}\left(e^{i\theta}\right)\right), \tag{8.24}$$

valuable for all θ with $|\theta| < 10c$, we deduce the desired inequality (8.17) as follows:

(8.25)

$$\left\{\begin{aligned} X^1_{1;c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right) &\geq X^1_{1;c,0,0,v_1+v(c)}\left(e^{i\theta}\right) + x_1 - x_1 \cdot c^{2+\alpha} \cdot K_{11} \\ &\geq \widetilde{g}\left(X'^{1}_{c,0,0,v_1+v(c)}\left(e^{i\theta}\right)\right) + x_1 - x_1 \cdot c^{2+\alpha} \cdot K_{11} \\ &\geq \widetilde{g}\left(X'^{1}_{c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right)\right) + x_1 - x_1 \cdot c \cdot K_{11} - x_1 \cdot c \cdot K_{12} \\ &> \widetilde{g}\left(X'^{1}_{c,x_1,0,v_1+v(c)}\left(e^{i\theta}\right)\right), \end{aligned}\right.$$

for all x_1 with $0 < x_1 \leq c^2$, all θ with $|\theta| < 10c$ and all c with $0 < c \leq c_1$, provided

$$c_1 \leq \frac{1/2}{K_{11} + K_{12}}. \tag{8.26}$$

This completes the proof of property $\mathbf{(8_1)}$.

Thirdly, let us prove property $\mathbf{(9_1)}$ in Case $\mathbf{(I_1)}$. The half-wedge $\mathcal{HW}_1^+$ is defined by the n inequalities of the last two lines of (5.38), where $a_2 + \cdots + a_n = 1$. For notational reasons, it will be convenient to set $a_1 := 1$ and to write the first inequality defining $\mathcal{HW}_1^+$ simply as $\sum_{j=1}^n a_j y_j > \psi(x, y')$.

Because $\Phi_c\left(\overline{\Delta}\backslash\partial^+\Delta\right)$ is contained in the open sector $\{\rho e^{i\theta} \in \overline{\Delta} : |\theta| < 10c,\ 1 - 10c < \rho < 1$, taking account of the definition (7.53) of $A^1_{x,v:c}(\zeta)$, in order to check property $\mathbf{(9_1)}$, it clearly suffices to show that $Z^1_{c,x,v_1+v(c)+v}\left(\left\{\rho e^{i\theta} \in \Delta :\ 1 - 10c < \rho < 1,\ |\theta| < 10c\right\}\right)$ is contained in $\mathcal{HW}_1^+$, which amounts to establish that for all x with $|x| \leq c^2$, all v with $|v| \leq c$, all $\rho e^{i\theta}$ with $1 - 10c < \rho < 1$ and with $|\theta| < 10c$, the following two collections of strict inequalities hold true

(8.27)

$$\left\{\begin{aligned} &\sum_{k=1}^n a_k Y^1_{j;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right) > \psi\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\ Y'^{1}_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right), \\ &\qquad Y^1_{j;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right) > \varphi_j\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\ Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right), \end{aligned}\right.$$

for $j = 2, \ldots, n$, provided c_1 is sufficiently small, where we use the notation (8.16).

We first treat the collection of $(n-1)$ strict inequalities in the second line of (8.27). First of all, by differentiating (7.8) twice with respect to θ, we obtain

$$\frac{\partial^2 Z^0_{c,x,v_1+v(c)+v}}{\partial\theta^2}\left(e^{i\theta}\right) = c \cdot \frac{\partial^2 \Psi}{\partial\theta^2}\left(e^{i\theta}\right) \cdot \left[v_1 + v(c) + v\right]. \tag{8.28}$$

Using the second estimate (7.36), we deduce that there exists a constant $K_{13} > 0$ such that

$$\left| \frac{\partial^2 Z^1_{c,x,v_1+v(c)+v}}{\partial \theta^2} \left(e^{i\theta}\right) \right| \leq c \cdot K_{13}. \tag{8.29}$$

Using the inequality (6.2), using (8.29), and then taking account of the inequalities $|\theta| < 10c$, $|x| \leq c^2$ and $|v| < c$, we deduce the following inequality

$$\left\{ \begin{aligned} \left| \frac{\partial Z^1_{c,x,v_1+v(c)+v}}{\partial \theta} \left(e^{i\theta}\right) - \frac{\partial Z^1_{c,0,v_1+v(c)}}{\partial \theta}(1) \right| & \leq c \cdot (|\theta| + |x| + |v|) \\ & \leq c^2 \cdot K_{14}, \end{aligned} \right. \tag{8.30}$$

for some constant $K_{14} > 0$. On the other hand, by differentiating (7.8) with respect to θ at $\theta = 0$ and applying the inequality (7.28), we obtain

$$\left| \frac{\partial Z^1_{c,0,v_1+v(c)}}{\partial \theta}(1) - c \cdot C_1 \cdot (0, 1, \ldots, 1) \right| \leq c^{2+\alpha} \cdot K_6, \tag{8.31}$$

where $C_1 = \frac{\partial \Psi}{\partial \theta}(1)$, as defined in (7.47). We remind that for every $\mathcal{C}^1$-smooth function Z on $\overline{\Delta}$ which is holomorphic in Δ, we have $i \frac{\partial}{\partial \theta} Z(e^{i\theta}) = -\frac{\partial}{\partial \rho} Z(e^{i\theta})$. Consequently, we deduce from (8.30) the following first (among three) interesting inequality

$$\left| -\frac{\partial Z^1_{c,x,v_1+v(c)+v}}{\partial \rho} \left(e^{i\theta}\right) - c \cdot C_1 \cdot (0, i, \ldots, i) \right| \leq c^2 \cdot K_{15}, \tag{8.32}$$

for some constant $K_{15} > 0$.

According to the definition (7.8), we may compute

$$\frac{\partial^2 Z^0_{c,x,v_1+v(c)+v}}{\partial \rho^2} \left(\rho e^{i\theta}\right) = c \cdot \frac{\partial^2 \Psi}{\partial \rho^2} \left(\rho e^{i\theta}\right) \cdot (v_1 + v(c) + v) \tag{8.33}$$

By reasoning as in the proof of Lemma 7.34, we may obtain an inequality similar to (7.36), with the second order partial derivative $\partial^2/\partial\theta^2$ replaced by the second order partial derivative $\partial^2/\partial\rho^2$. Putting this together with (8.33), we deduce that there exists a constant $K_{16} > 0$ such that

$$\left| \frac{\partial^2 Z^1_{c,x,v_1+v(c)+v}}{\partial \rho^2} \left(\rho e^{i\theta}\right) \right| \leq c \cdot 2K_{16}, \tag{8.34}$$

for some constant $K_{16} > 0$. Applying then Taylor's integral formula $F(\rho) = F(1) + (\rho - 1) \cdot \partial_\rho F(1) + \int_1^\rho (\rho - \widetilde{\rho}) \cdot \partial_\rho \partial_\rho F(\widetilde{\rho}) \cdot d\widetilde{\rho}$ to the functions $F(\rho) := Y^1_{k;c,x,v_1+v(c)+v} \left(\rho e^{i\theta}\right)$ for $k = 1, \ldots, n$, we deduce the second interesting collection of inequalities

$$\left\{ \begin{aligned} & \Big| Y^1_{k;c,x,v_1+v(c)+v} \left(\rho e^{i\theta}\right) - Y^1_{k;c,x,v_1+v(c)+v} \left(e^{i\theta}\right) - \\ & \qquad - (\rho - 1) \cdot \frac{\partial Y^1_{k;c,x,v_1+v(c)+v}}{\partial \rho} \left(e^{i\theta}\right) \Big| \leq (1-\rho)^2 \cdot c \cdot K_{16}, \end{aligned} \right. \tag{8.35}$$

for $k = 1, \ldots, n$.

On the other hand, thanks to the normalizations of the functions $\varphi_j(x, y_1)$ given in (5.40), namely $\varphi_j(0) = \partial_{x_k} \varphi_j(0) = \partial_{y_1} \varphi_j(0) = 0$, $j = 2, \ldots, n$, $k = 1, \ldots, n$,

we see that, possibly after increasing the constant $K_1 > 0$ of Lemma 6.4, we have inequalities of the form

(8.36)
$$\left\{\begin{aligned}&\sum_{k=1}^{n}|\varphi_{j,x_k}(x,y_1)|+|\varphi_{j,y_1}(x,y_1)|\leq(|x|+|y_1|)\cdot K_1,\\&|\varphi_j(x,y_1)-\varphi_j\left(\widetilde{x},\widetilde{y}_1\right)|\leq(|x-\widetilde{x}|+|y_1-\widetilde{y}_1|)\cdot\left(\sum_{k=1}^{n}\sup_{|x|,\,|y_1|\leq c\cdot K_2}|\varphi_{j,x_k}(x,y_1)|+\right.\\&\qquad\qquad\left.+\sup_{|x|,\,|y_1|\leq c\cdot K_2}|\varphi_{j,y_1}(x,y_1)|\right),\end{aligned}\right.$$

for $j=2,\dots,n$, provided $|x|,\,|\widetilde{x}|,\,|y_1|,\,|\widetilde{y_1}|\leq c\cdot K_2$. On the other hand, computing $\frac{\partial Z^0_{c,x,v_1+v(c)+v}}{\partial\rho}\left(\rho e^{i\theta}\right)$ in (7.8), using (7.25), (7.28) and an inequality of the form (6.2), we deduce that there exists a constant $K_{17}>0$ such that

$$\left|Z^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)-Z^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right|\leq(1-\rho)\cdot c\cdot K_{17}. \tag{8.37}$$

Finally, using the inequality $|Z^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)|\leq c\cdot K_2$ obtained in (7.25), using the collection of inequalities (8.36) and using the inequality (8.37), we may deduce the third (and last) interesting inequality for $j=2,\dots,n$:

(8.38)
$$\left\{\begin{aligned}&\left|\varphi_j\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\,Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right)-\right.\\&\qquad\left.-\varphi_j\left(X^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right),\,Y^1_{1;c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right)\right|\leq\\&\left(\left|X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)-X^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right|+\right.\\&\qquad\left.+\left|Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)-Y^1_{1;c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right|\right)\cdot\\&\qquad\cdot\left(\sum_{k=1}^{n}\sup_{|x|,\,|y_1|\leq c\cdot K_2}|\varphi_{j,x_k}(x,y_1)|+\sup_{|x|,\,|y_1|\leq c\cdot K_2}|\varphi_{j,y_1}(x,y_1)|\right)\leq\\&\leq(1-\rho)\cdot c^2\cdot K_{18},\end{aligned}\right.$$

for some constant $K_{18}>0$.

We can now complete the proof of the collection of inequalities in the second line of (8.27). As before, let c with $0<c\leq c_1$, let ρ with $10c<\rho<1$, let θ with $|\theta|<10c$, let x with $|x|\leq c^2$, let v with $|v|\leq c$ and let $j=2,\dots,n$. Starting with (8.35), using (8.32), using the fact that $Z^1_{c,x,v_1+v(c)+v}\left(\partial^+\Delta\right)\subset M^1\subset M$ and using (8.38), we

have

(8.39)
$$\begin{cases} Y^1_{j;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right) \geq \\ \quad \geq Y^1_{j;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) + \\ \quad\quad + (\rho-1)\cdot \dfrac{\partial Y^1_{j;c,x,v_1+v(c)+v}}{\partial\rho}\left(e^{i\theta}\right) - (1-\rho)^2\cdot c\cdot K_{16} \geq \\ \quad \geq Y^1_{j;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) + (1-\rho)\cdot c\cdot C_1 - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} \\ \quad = \varphi_j\left(X^1_{1;c,x,v_1+v(c)+v}\left(e^{i\theta}\right),\, Y^1_{1;c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right) + (1-\rho)\cdot c\cdot C_1 - \\ \qquad\qquad\qquad - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} \\ \quad \geq \varphi_j\left(X^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) + (1-\rho)\cdot c\cdot C_1 - \\ \qquad\qquad - (1-\rho)\cdot c^2\cdot K_{15} - (1-\rho)^2\cdot c\cdot K_{16} - (1-\rho)\cdot c^2\cdot K_{18} \\ \quad \geq \varphi_j\left(X^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) + (1-\rho)\cdot c\cdot [C_1 - \\ \qquad\qquad\qquad - c\cdot K_{15} - 10c\cdot K_{16} - c\cdot K_{18}] \\ \quad \geq \varphi_j\left(X^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) + (1-\rho)\cdot c\cdot \dfrac{C_1}{2} \\ \quad > \varphi_j\left(X^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y^1_{1;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right), \end{cases}$$

provided that

$$c_1 \leq \frac{C_1/2}{K_{15}+10K_{16}+K_{18}}. \tag{8.40}$$

This yields the collection of inequalities in the second line of (8.27).

For the first inequality (8.27), we proceed similarly. Recall that $v_1 = (0,1,\dots,1)$, that $a_1 = 1$ and that $a_2 + \cdots + a_n = 1$. Since $Z^1_{c,x,v_1+v(c)+v}(\partial^+\Delta) \subset M^1 \subset N^1$, we have for all θ with $|\theta| \leq \frac{\pi}{2}$ the following relation

(8.41)
$$\sum_{k=1}^n a_k\, Y^1_{k;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) = \psi\left(X^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right).$$

Using that ψ vanishes to order one at the origin by the normalization conditions (5.40) and proceeding as in the previous paragraph concerning the functions φ_j, we obtain an inequality similar to (8.38):

(8.42)
$$\begin{cases} \Big|\psi\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) - \\ \qquad - \psi\left(X^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right)\Big| \leq (1-\rho)\cdot c^2\cdot K_{19}, \end{cases}$$

for some constant $K_{19} > 0$.

As before, let c with $0 < c \leq c_1$, let ρ with $10c < \rho < 1$, let θ with $|\theta| < 10c$, let x with $|x| \leq c^2$ and let v with $|v| \leq c$. Using then (8.35), (8.32), (8.41) and (8.42), we

deduce the desired strict inequality

(8.43)

$$\left\{\begin{aligned}
&\sum_{k=1}^{n} a_k\, Y^1_{k;c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right) \geq \sum_{k=1}^{n} a_k\, Y^1_{k;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) + \\
&\qquad + (1-\rho)\left[\sum_{k=1}^{n} a_k \left(-\frac{\partial Y^1_{k;c,x,v_1+v(c)+v}}{\partial \rho}\left(e^{i\theta}\right)\right)\right] - (1-\rho)^2 \cdot c \cdot \left(\sum_{k=1}^{n} a_k\right) K_{16} \\
&\qquad \geq \sum_{k=1}^{n} a_k\, Y^1_{k;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) + (1-\rho)\left[\sum_{j=2}^{n} a_j \cdot c \cdot C_1 - \sum_{k=1}^{n} a_k \cdot c^2 \cdot K_{15}\right] - \\
&\qquad\qquad - (1-\rho)^2 \cdot c \cdot 2K_{16} \\
&\qquad \geq \sum_{k=1}^{n} a_k\, Y^1_{k;c,x,v_1+v(c)+v}\left(e^{i\theta}\right) + (1-\rho)\cdot c \cdot C_1 - \\
&\qquad\qquad\qquad - (1-\rho)\cdot c^2 \cdot 2K_{15} - (1-\rho)^2 \cdot c \cdot 2K_{16} \\
&\qquad = \psi\left(X^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(e^{i\theta}\right)\right) + \\
&\qquad\qquad\qquad + (1-\rho)\cdot c \cdot \left[C_1 - c \cdot 2K_{15} - 10c \cdot 2K_{16}\right] \\
&\qquad \geq \psi\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) + \\
&\qquad\qquad\qquad + (1-\rho)\cdot c \cdot \left[C_1 - c \cdot 2K_{15} - 10c \cdot 2K_{16} - c \cdot K_{19}\right] \\
&\qquad \geq \psi\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right) + (1-\rho)\cdot c \cdot \frac{C_1}{2} \\
&\qquad > \psi\left(X^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right),\, Y'^1_{c,x,v_1+v(c)+v}\left(\rho e^{i\theta}\right)\right),
\end{aligned}\right.$$

provided

(8.44)
$$c_1 \leq \frac{C_1/2}{2K_{15} + 20K_{16} + K_{19}}$$

This yields the first inequality of (8.27) and completes the proof of $\mathbf{(9_1)}$ in Case $\mathbf{(I_1)}$.

The proof of Lemma 8.3 is complete, because by following the normalizations of Lemma 5.37 and by formulating analogous inequalities, cases $\mathbf{(I_2)}$ and **(II)** are achieved in a totally similar way. □

§9. End of proof of Theorem 1.2': application of the continuity principle

9.1. Preliminary. In this section, we shall now complete the proof of Proposition 5.12 (at last!), hence the proof of Theorem 3.19 **(i)** (which is a direct consequence of Proposition 5.12, as was explained in Section 5) and hence also the proof of Theorem 1.2', modulo supplementary arguments postponed to §9.27 below. By means of a deformation $A^1_{x,v,u:c}(\zeta)$ (we add a real parameter u) of the family of analytic discs $A^1_{x,v:c}(\zeta)$ satisfying properties $\mathbf{(1_1)}$ to $\mathbf{(9_1)}$ of Lemmas 7.12 and 8.3, and by means of the continuity principle, we shall show that, in Cases $\mathbf{(I_1)}$ and $\mathbf{(I_2)}$, there exists a local wedge $\mathcal{W}_{p_1}$ of edge M at p_1 to which all holomorphic functions in $\mathcal{O}\left(\Omega \cup \mathcal{HW}_1^+\right)$ extend holomorphically and we shall show that in Case **(II)**, there exists a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$ to which all holomorphic functions in $\mathcal{O}\left(\Omega \cup \mathcal{W}_2\right)$ extend holomorphically. To organize this last main step of the proof of Proposition 5.12, we shall consider jointly Cases $\mathbf{(I_1)}$, $\mathbf{(I_2)}$ and then afterwards Case **(II)** separately in §9.22 below.

9.2. Isotopies of analytic discs and continuity principle. To begin with, we shall formulate a convenient version of the continuity principle. If $E \subset \mathbb{C}^n$ is an arbitrary subset, we denote by

$$\mathcal{V}_{\mathbb{C}^n}(E, \rho) = \bigcup_{p \in E} \{z \in \mathbb{C}^n : |z - p| < \rho\} \tag{9.3}$$

the union of polydiscs of radius $\rho > 0$ centered at points of E. We then have the following lemma, extracted from [M2], which applies to families of analytic discs $A_\tau(\zeta)$ which are embeddings of $\overline{\Delta}$ into $\mathbb{C}^n$:

Lemma 9.4. ([M2], Proposition 3.3) *Let $\mathcal{D}$ be a nonempty domain in $\mathbb{C}^n$ and let $A_\tau : \overline{\Delta} \to \mathbb{C}^n$ be a one-parameter family of analytic discs, where $\tau \in \mathbb{R}$ satisfies $0 \leq \tau \leq 1$. Assume that there exist constants c_τ and C_τ with $0 < c_\tau < C_\tau$ such that*

$$c_\tau |\zeta_1 - \zeta_2| < |A_\tau(\zeta_1) - A_\tau(\zeta_2)| < C_\tau |\zeta_1 - \zeta_2|, \tag{9.5}$$

for all distinct points $\zeta_1, \zeta_2 \in \overline{\Delta}$ and all $0 \leq \tau \leq 1$. Assume that $A_1(\overline{\Delta}) \subset \mathcal{D}$, set $\rho_\tau := \inf\{|t - A_\tau(\zeta)| : t \in \partial\mathcal{D}, \zeta \in \partial\Delta\}$, namely ρ_τ is the polydisc distance between $A_\tau(\partial\Delta)$ and $\partial\mathcal{D}$, assume $\rho_\tau > 0$ for all τ with $0 \leq \tau \leq 1$, and set $\sigma_\tau := \rho_\tau c_\tau / 2C_\tau$. Then for every holomorphic function $f \in \mathcal{O}(\mathcal{D})$, and for every $\tau \in [0,1]$, there exists a holomorphic function $F_\tau \in \mathcal{O}\left(\mathcal{V}_{\mathbb{C}^n}(A_\tau\left(\overline{\Delta}\right), \sigma_\tau)\right)$ such that $F_\tau = f$ in $\mathcal{V}_{\mathbb{C}^n}(A_\tau(\partial\Delta), \sigma_\tau) \subset \mathcal{D}$.

Two analytic discs $A', A'' : \overline{\Delta} \to \mathbb{C}^n$ which are of class $\mathcal{C}^1$ over $\overline{\Delta}$ and holomorphic in Δ and which are both *embeddings* of $\overline{\Delta}$ into $\mathbb{C}^n$ are said to be *analytically isotopic* if there exists a $\mathcal{C}^1$-smooth family of analytic discs $A_\tau : \overline{\Delta} \to \mathbb{C}^n$ which are of class $\mathcal{C}^1$ over $\overline{\Delta}$ and holomorphic in Δ such that $A_0 = A'$, such that $A_1 = A''$ and such that A_τ is an *embedding* of $\overline{\Delta}$ into $\mathbb{C}^n$ for all τ with $0 \leq \tau \leq 1$.

9.6. Translations of M^1 in M. According to Lemma 5.37, in Case $(\mathbf{I_1})$, the one-codimensional submanifold $M^1 \subset M$ is given by the equations $y' = \varphi'(x, y_1)$ and $x_1 = g(x')$. If $u \in \mathbb{R}$ is a small real parameter, we may define a "translation" M^1_u of M^1 in M by the n equations

$$M^1_u : \quad y' = \varphi'(x, y_1), \quad x_1 = g(x') + u. \tag{9.7}$$

Clearly, we have $M^1_0 \equiv M^1$, we have $M^1_u \subset (M^1)^+$ if $u > 0$ and we have $M^1_u \subset (M^1)^-$ if $u < 0$. We may perturb the family of analytic discs $Z^d_{c,x,v}(\zeta)$ attached to M^1 and satisfying Bishop's equation (7.18) by requiring that it is attached to M^1_u. We then obtain a new family of analytic discs $Z^d_{c,x,v,u}(\zeta)$ which is half-attached to M^1_u and which is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all variables (c, x, v, u, ζ), thanks to the stability under perturbation of the solutions to Bishop's. For $u = 0$, this solution coincides with the family $Z^d_{c,x,v}(\zeta)$ constructed in §7.13. Using a similar definition as in (7.53), namely setting $A^1_{x,v,u:c}(\zeta) := Z^1_{c,x,v_1+v(c)+v,u}\left(\Phi_c(\zeta)\right)$, we obtain a new family of analytic discs which coincides, for $u = 0$, with the family of analytic discs $A^1_{x,v:c}(\zeta)$ of Lemmas 7.12 and 8.3. Similarly, in Case $(\mathbf{I_2})$, taking account of the normalizations stated in Lemma 5.37, we can also construct an analogous family of analytic discs $A^1_{x,v,u:c}(\zeta)$. From now on, we shall fix the scaling parameter c with $0 < c \leq c_1$, so that the nine properties $(\mathbf{1_1})$ to $(\mathbf{9_1})$ of Lemmas 7.12 and 8.3 are satisfied.

9.8. Definition of a local wedge of edge M at p_1 in Cases ($\mathbf{I_1}$) and ($\mathbf{I_2}$). First of all, in Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**, we shall restrict the variation of the parameter v to a certain $(n-2)$-dimensional linear subspace V_2 of $T_{p_1}\mathbb{R}^n \cong \mathbb{R}^n$ as follows. By hypothesis, the vector v_1 does not belong to the characteristic direction $T_{p_1}M^1 \cap T^c_{p_1}M$, so the real vector space $(\mathbb{R}\cdot v_1)\oplus\big(T_{p_1}M^1 \cap T^c_{p_1}M\big) \subset T_{p_1}M^1$ is 2-dimensional. We choose an arbitrary $(n-2)$-dimensional real vector subspace $V_2 \subset T_{p_1}M^1$ which is a supplementary in $T_{p_1}M^1$ to $(\mathbb{R}\cdot v_1)\oplus\big(T_{p_1}M^1 \cap T^c_{p_1}M\big)$ and we shall let the parameter v vary only in V_2.

From the rank properties **($\mathbf{5_1}$)** and **($\mathbf{6_1}$)** of Lemma 7.12 and from the definition of V_2, we deduce that there exists $\varepsilon > 0$ small enough with $\varepsilon << c^2$ such that the mapping

$$(x, v, u, \rho) \longmapsto A^1_{x,v,u:c}(\rho) \tag{9.9}$$

is a *one-to-one immersion* from the open set $\{(x, v, u, \rho) \in \mathbb{R}^n \times V_2 \times \mathbb{R} \times \mathbb{R} : \ |x| < \varepsilon, \ |v| < \varepsilon, \ |u| < \varepsilon, \ 1-\varepsilon < \rho < 1\}$ into $\mathbb{C}^n$. This property will be important for uniqueness of the holomorphic extension in our application of the continuity principle to be conducted in Lemma 9.20 below. Furthermore, shrinking $\varepsilon > 0$ if necessary, we can insure that the open subset

$$\begin{cases} \mathcal{W}_{p_1} := \big\{A^1_{x,v,u:c}(\rho) \in \mathbb{C}^n : (x, v, u, \rho) \in \mathbb{R}^n \times V_2 \times \mathbb{R} \times \mathbb{R}, \\ \qquad\qquad\qquad\qquad |x| < \varepsilon, \ |v| < \varepsilon, \ |u| < \varepsilon, \ 1-\varepsilon < \rho < 1\big\} \end{cases} \tag{9.10}$$

is a local wedge of edge M at (p_1, Jv_1), with $\mathcal{W}_{p_1} \cap M = \emptyset$.

Let the closed subset C with $p_1 \in C$ and $C\backslash\{p_1\} \subset (H^1)^-$, let the neighborhood Ω of $M\backslash C$ in $\mathbb{C}^n$, let the half-wedge $\mathcal{HW}^+_{p_1}$ be as in Proposition 5.12, and let the sub-half-wedge $\mathcal{HW}^+_1 \subset \mathcal{HW}^+_{p_1}$ be as in §5.14 and Lemma 5.37. In Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**, we shall consider the envelope of holomorphy of the open subset $\Omega \cup \mathcal{HW}^+_1$. We shall prove in the next paragraphs that, after possibly shrinking it a little bit, its envelope of holomorphy contains the wedge $\mathcal{W}_{p_1}$.

9.11. Boundaries of analytic discs. Since we want to apply the continuity principle Lemma 9.4, we have to show that most discs $A^1_{x,v,u:c}(\zeta)$ have their boundaries in $\Omega \cup \mathcal{HW}^1_1+$. To this aim, it will be useful to decompose the boundary $\partial\Delta$ in three closed parts $\partial\Delta = \partial^1\Delta \cup \partial^2\Delta \cup \partial^3\Delta$, where

$$\begin{cases} \partial^1\Delta := \big\{e^{i\theta} \in \partial\Delta : \ |\theta| \leq \pi/2 - \varepsilon\big\} \subset \partial^+\Delta, \\ \partial^2\Delta := \big\{e^{i\theta} \in \partial\Delta : \ \pi/2 + \varepsilon \leq |\theta| \leq \pi\big\} \subset \partial^-\Delta, \\ \partial^3\Delta := \big\{e^{i\theta} \in \partial\Delta : \ \pi/2 - \varepsilon \leq |\theta| \leq \pi/2 + \varepsilon\big\} \subset \partial\Delta, \end{cases} \tag{9.12}$$

where ε with $0 < \varepsilon << c^2$ is as in §9.8 just above. This decomposition is illustrated in the left hand side of FIGURE 17 below. Next, we observe that the two points $A^1_{0,0,0:c}(i)$ and $A^1_{0,0,0:c}(-i)$ belong to $(H^1)^- \subset M\backslash C \subset \Omega$, hence there exists a fixed open neighborhood of these two points which is contained in Ω. We shall denote by ω^3 such a (disconnected) neighborhood, for instance the union of two small open polydiscs centered at these two points. To proceed further, we need a crucial geometric information about the boundaries of the analytic discs $A^1_{x,v,u:c}(\zeta)$ with $u \neq 0$.

Lemma 9.13. *Under the assumptions of Cases* **($\mathbf{I_1}$)** *and* **($\mathbf{I_2}$)** *of Proposition 5.12, after shrinking $\varepsilon > 0$ if necessary, then*

$$A^1_{x,v,u:c}\,(\partial\Delta) \subset \Omega \cup \mathcal{HW}^+_1, \tag{9.14}$$

for all x with $|x| < \varepsilon$, for all v with $|v| < \varepsilon$ and for all nonzero *$u \neq 0$ with $|u| < \varepsilon$.*

Proof. Firstly, since $A^1_{0,0,0:c}(\pm i) \in \omega^3$, it follows just by continuity of the family $A^1_{x,v,u:c}(\zeta)$ that, after possibly shrinking $\varepsilon > 0$, the closed arc $A^1_{x,v,u:c}\left(\partial^3\Delta\right)$ is contained in ω^3, for all x with $|x| < \varepsilon$, for all v with $|v| < \varepsilon$ and for all u with $|u| < \varepsilon$. Secondly, since $A^1_{0,0,0:c}(\partial^2\Delta) \subset A^1_{0,0,0:c}\left(\partial^-\Delta\backslash\{i,-i\}\right) \subset \mathcal{HW}^+_1$, then by property **($\mathbf{9_1}$)** of Lemma 8.3, it follows just by continuity of the family $A^1_{x,v,u:c}(\zeta)$ that, after possibly shrinking $\varepsilon > 0$, the closed arc $A^1_{x,v,u:c}\left(\partial^2\Delta\right)$ is contained in $\mathcal{HW}^+_1$, for all x with $|x| < \varepsilon$, for all v with $|v| < \varepsilon$ and for all u with $|u| < \varepsilon$. Thirdly, it follows from the inclusion $A^1_{x,v,u:c}\left(\partial^1\Delta\right) \subset A^1_{x,v,u:c}\left(\partial^+\Delta\right) \subset M^1_u$ and from the inclusion $M^1_u \subset \Omega$ for all $u \neq 0$ that, after possibly shrinking $\varepsilon > 0$, the closed arc $A^1_{x,v,u:c}\left(\partial^1\Delta\right)$ is contained in Ω, for all x with $|x| < \varepsilon$, for all v with $|v| < \varepsilon$ and for all u with $|u| < \varepsilon$ and $u \neq 0$. This completes the proof of Lemma 9.13. □

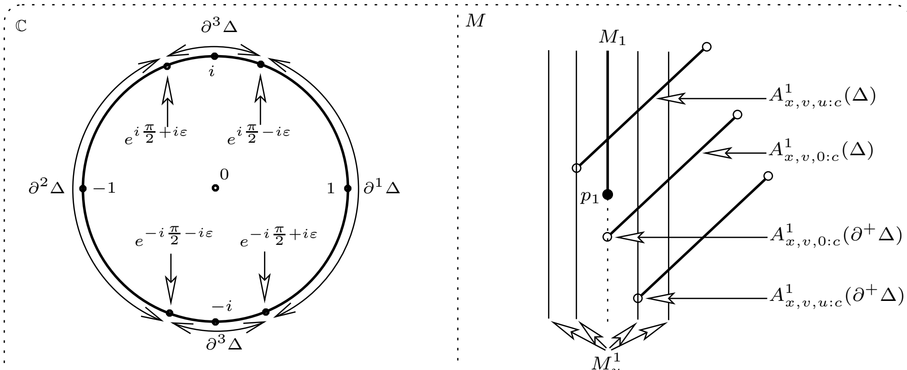

FIGURE 17: DECOMPOSITION OF $\partial\Delta$ AND ISOTOPIES OF THE ANALYTIC DISCS $A_{x,v,u:c}(\zeta)$

In addition to this lemma, we notice that it follows immediately from properties **($\mathbf{8_1}$)** and **($\mathbf{9_1}$)** of Lemma 8.3 that $A^1_{x,0,0:c}(\overline{\Delta})$ is contained in $\Omega \cup \mathcal{HW}^+_1$ for all x with $|x| < \varepsilon$ such that $A^1_{x,0,0:c}(1) \in T^1 \cap (H^1)^+$. This property and Lemma 9.13 are illustrated in the right hand side of FIGURE 17 just above.

9.15. Analytic isotopies. Next, in Case **($\mathbf{I_1}$)**, we fix some $x_0 = (x_{1;0}, 0, \dots, 0) \in \mathbb{R}^n$ with $0 < x_{1;0} < \varepsilon$. Then $A^1_{x_0,0,0:c}(1) = x_0 + ih(x_0)$ belongs to $T^1 \cap (H^1)^+$. Analogously, in Cases **($\mathbf{I_2}$)**, we fix some $x_0 = (0, \dots, 0, x_{n;0}) \in \mathbb{R}^n$ with $0 < x_{n;}0 < \varepsilon$. Then in this second case, the point $A^1_{x_0,0,0:c}(1) = x_0 + ih(x_0)$ also belongs to $T^1 \cap (H^1)^+$. We fix the disc $A^1_{x_0,0,0:c}(\zeta)$, which satisfies $A^1_{x_0,0,0:c}(\overline{\Delta}) \subset \Omega \cup \mathcal{HW}^+_1$.

Lemma 9.16. *In Cases* ($\mathbf{I_1}$) *and* **($\mathbf{I_2}$)**, *every disc* $A^1_{x,v,u:c}(\zeta)$ *with* $|x| < \varepsilon$, $|v| < \varepsilon$, $|u| < \varepsilon$ *and* $u \neq 0$ *is analytically isotopic to the disc* $A^1_{x_0,0,0:c}(\zeta)$, *with the boundaries of the analytic discs of the isotopy being all contained in* $\Omega \cup \mathcal{HW}^+_1$.

Proof. Indeed, since the set $\{u = 0\}$ is a hyperplane, there clearly exists a $\mathcal{C}^{2,\alpha-0}$-smooth curve $\tau \mapsto (x(\tau), v(\tau), u(\tau))$ in the parameter space which joins a given arbitrary point (x^*, v^*, u^*) with $u^* \neq 0$ to the point $(x_0, 0, 0)$ without meeting the hyperplane $\{u = 0\}$, except at its endpoint $(x_0, 0, 0)$. According to the previous Lemma 9.13, each boundary $A^1_{x(\tau),v(\tau),u(\tau):c}(\partial\Delta)$ is then automatically contained in $\Omega \cup \mathcal{HW}^+_1$, which completes the proof. □

9.17. holomorphic extension to a local wedge of edge M at p_1. In Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**, we define the following $\mathcal{C}^{2,\alpha-0}$-smooth connected hypersurface of $\mathcal{W}_{p_1}$:

$$\begin{cases} \mathcal{M}_{p_1} := \big\{A^1_{x,v,0:c}(\rho) : \ (x,v,\rho) \in \mathbb{R}^n \times V_2 \times \mathbb{R}, \\ \qquad\qquad\qquad |x| < \varepsilon, \ |v| < \varepsilon, \ 1-\varepsilon < \rho < \varepsilon\big\}, \end{cases} \tag{9.18}$$

together with the following closed subset of $\mathcal{M}_{p_1}$:

(9.19)

$$\begin{cases} \mathcal{C}_{p_1} := \big\{A^1_{x,v,0:c}(\rho) : \ (x,v,\rho) \in \mathbb{R}^n \times V_2 \times \mathbb{R}, \\ \qquad\qquad\qquad A^1_{x,v,0:c}(\partial^+\Delta) \cap C = \emptyset, \ |x| < \varepsilon, \ |v| < \varepsilon, \ 1-\varepsilon < \rho < \varepsilon\big\}. \end{cases}$$

Since $A^1_{x,0,0:c}(\partial^+\Delta)$ is contained in $(H^1)^+$ for all x such that $A^1_{x,0,0}(1) \in T^1 \cap (H^1)^+$, the closed subset $\mathcal{C}_{p_1}$ of $\mathcal{M}_{p_1}$ is a *proper* closed subset of $\mathcal{M}_{p_1}$. The following figure provides a geometric illustration.

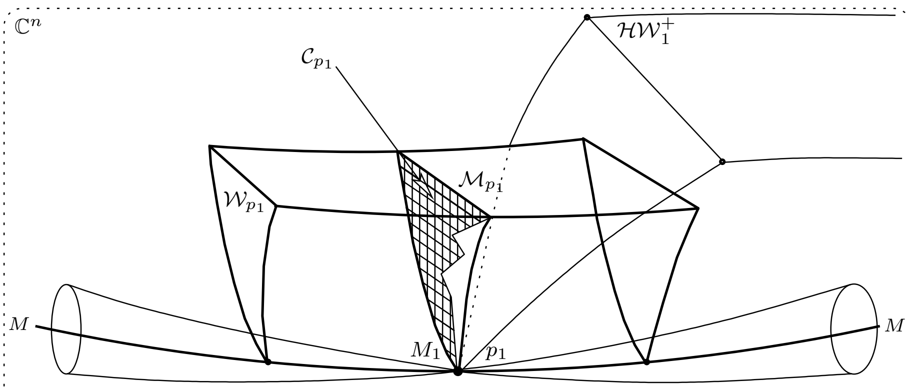

FIGURE 18: THE PROPER CLOSED SUBSET $\mathcal{C}_{p_1}$ OF THE HYPERSURFACE $\mathcal{M}_{p_1} \subset \mathcal{W}_{p_1}$

We can now state the main lemma of this section, which will complete the proof of Proposition 5.12 in Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**.

Lemma 9.20. *After possibly shrinking Ω in a small neighborhood of p_1 and after possibly shrinking $\varepsilon > 0$, the set $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ is connected and for every holomorphic function $f \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_1^+\right)$, there exists a holomorphic function $F \in \mathcal{O}\left(\Omega \cup \mathcal{HW}_1^+ \cup \mathcal{W}_{p_1}\right)$ such that $F|_{\Omega \cup \mathcal{HW}_1^+} = f$.*

Proof. Remind that $\varepsilon << c^2$ and remind that the wedge $\mathcal{W}_{p_1}$ with $\mathcal{W}_{p_1} \cap M = \emptyset$ in the two cases is of size $\mathrm{O}(\varepsilon)$. Since C is contained in $(H^1)^- \cup \{p_1\} \subset M^1$, we observe that $M \backslash C$ is locally connected at p_1. Since the half-wedge $\mathcal{HW}_1^+$ defined in Lemma 5.37 by simple inequalities is of size $\mathrm{O}(\delta_1)$, if moreover $\varepsilon << \delta_1$, after shrinking Ω if necessary in a smal neighborhood of p_1 whose size is $\mathrm{O}(\varepsilon)$, it follows that we can assume that $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ is connected. However, in FIGURE 18 above, because we draw M as if it were one-dimensional, the intersection $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ appears to be disconnected, which is a slight incorrection.

Let f be an arbitrary holomorphic function in $\mathcal{O}\left(\Omega \cup \mathcal{HW}_1^+\right)$. Thanks to the isotopy Lemma 9.16 and thanks to the continuity principle Lemma 9.4, we deduce that f extends holomorphically to a neighborhood in $\mathbb{C}^n$ of every disc $A_{x,v,u:c}(\overline{\Delta})$ whose boundary $A_{x,v,u:c}(\partial\Delta)$ is contained in $\Omega \cup \mathcal{HW}_1^+$. Using the fact that the mapping (9.9) is one-to one, we deduce that we can extend f uniquely by means of Cauchy's formula at points of

the form $A^1_{x,v,u:c}(\rho)$ with such values $|x| < \varepsilon$, $|v| < \varepsilon$, $|u| < \varepsilon$ and $1 - \varepsilon < \rho < \varepsilon$ for which $A^1_{x,v,u:c}(\partial\Delta) \subset \Omega \cup \mathcal{HW}_1^+$, simply as follows

$$f\left(A^1_{x,v,u:c}(\rho)\right) := \int_{\partial\Delta} \frac{f\left(A^1_{x,v,u:c}(\widetilde{\zeta})\right)}{\widetilde{\zeta} - \rho}\, d\widetilde{\zeta}. \tag{9.21}$$

With this definition, we extend f holomorphically and uniquely to the domain $\mathcal{W}_{p_1}\backslash\mathcal{C}_{p_1}$, where $\mathcal{C}_{p_1}$ is the proper closed subset, defined by (9.21), of the $\mathcal{C}^{2,\alpha-0}$-smooth hypersurface $\mathcal{M}_{p_1}$ defined by (9.20). Let $F \in \mathcal{O}\left(\mathcal{W}_{p_1}\backslash\mathcal{C}_{p_1}\right)$ denote this holomorphic extension. Since $\mathcal{W}_{p_1} \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ is connected, it follows that $\left[\mathcal{W}_{p_1}\backslash\mathcal{C}_{p_1}\right] \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ is also connected. By Lemma 9.4, the function f and its holomorphic extension F coincide in a neighborhood of every boundary $A^1_{x,v,u:c}(\partial\Delta)$ which is contained in the domain $\Omega \cup \mathcal{HW}_1^+$. From the analytic continuation principle, we deduce that there exists a well-defined function, still denoted by F, which is holomorphic in $\left[\mathcal{W}_{p_1}\backslash\mathcal{C}_{p_1}\right] \cap \left[\Omega \cup \mathcal{HW}_1^+\right]$ and which extends f, namely $F|_{\Omega\cup\mathcal{HW}_1^+} = f$.

To conclude the proof of Proposition 5.12 in Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$, it suffices to extend F holomorphically through the closed subset $\mathcal{C}_{p_1}$ of the connected hypersurface $\mathcal{M}_{p_1}$ in the domain $\mathcal{W}_{p_1} \subset \mathbb{C}^n$. Since $n \geq 2$, we notice that we are exactly in the situation of Theorem 1.4 in the special, much simpler case where the generic submanifold M is replaced by a domain of $\mathbb{C}^n$. It may therefore appear to be quite satisfactory to have reduced Theorem 1.2' to the CR dimension ≥ 2 version Theorem 1.4, in an open subset of $\mathbb{C}^n$ (notice however that Theorem 1.4 as well as Theorems 1.2 and 1.2' are stated in positive codimension, since the case where M is replaced by a domain of $\mathbb{C}^n$ is relatively trivial in comparison). The removability of such a proper closed subset contained in a connected hypersurface of a domain in $\mathbb{C}^n$ is known, follows from [J4] and is explicitely stated and proved as Lemma 2.10, p. 842 in [MP1]. However, we shall still provide another different geometric proof of this simple removability result in Lemma 10.10 below, using fully the techniques developed in the previous sections.

The proofs of Lemma 9.20 together with Cases $(\mathbf{I_1})$ and $(\mathbf{I_2})$ of Proposition 5.12 are complete now. □

9.22. End of proof of Proposition 5.12 in Case (II). According to Lemma 5.37, in Case **(II)**, the one-codimensional totally real submanifold $M^1 \subset M$ is given by the equations $y' = \varphi'(x, y_1)$ and $x_n = g(x'')$. If $u \in \mathbb{R}$ is a small real parameter, we may define a "translation" M^1_u of M^1 in M by the equations

$$y' = \varphi'(x, y_1), \qquad x_n = g\left(x''\right) + u. \tag{9.23}$$

Similarly as in §9.6, we may construct a family of analytic discs $A^1_{x,v,u:c}(\zeta)$ half-attached to M^1_u. We then we fix a small scaling parameter c with $0 < c \leq c_1$ so that properties $(\mathbf{1_1})$ to $(\mathbf{9_1})$ of Lemmas 7.12 and 8.3 hold true. Similarly as in §9.8, we shall restrict the variation of the parameter v to an arbitrary $(n-1)$-dimensional subspace V_1 of $T_{p_1}M^1 \cong \mathbb{R}^n$ which is supplementary to the real line $\mathbb{R}\cdot v_1$ in $T_{p_1}M^1$. If $\varepsilon > 0$ is small enough with $\varepsilon << c^2$, it follows that the mapping

$$(x, v, u, \rho) \longmapsto A^1_{x,v,u:c}(\rho) \tag{9.24}$$

is a *one-to-one immersion* from the open set $\{(x, v, \rho) \in \mathbb{R}^n \times V_1 \times \mathbb{R} \times \mathbb{R} : \ |x| < \varepsilon,\ |v| < \varepsilon,\ 1 - \varepsilon < \rho < 1\}$ into $\mathbb{C}^n$. Thanks to the choice of the linear subspace V_1,

shrinking $\varepsilon > 0$ if necessary, it follows that for every u with $|u| < \varepsilon$, the open subset

$$\text{(9.25)} \qquad \begin{cases} \mathcal{W}_u^1 := \big\{ A^1_{x,v,u:c}(\rho) \in \mathbb{C}^n : (x,v,\rho) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \\ \qquad\qquad\qquad\qquad |x| < \varepsilon, \ |v| < \varepsilon, \ 1-\varepsilon < \rho < 1 \big\} \end{cases}$$

is a local wedge of edge M_u^1. Clearly, this wedge $\mathcal{W}_u^1$ depends $\mathcal{C}^{2,\alpha-0}$-smoothly with respect to u.

Using the fact that in Case **(II)** we have

$$\text{(9.26)} \qquad \frac{\partial A^1_{0,0,0:c}}{\partial \theta}(1) = v_1 = (1,0,\dots,0) \in T_{p_1}M^1 \cap T^c_{p_1}M,$$

one can prove that Lemma 9.13 holds true with $\mathcal{HW}_1^+$ replaced by $\mathcal{W}_2$ in (9.14) and also that Lemma 9.16 holds true, again with $\mathcal{HW}_1^+$ replaced by $\mathcal{W}_2$. Similarly as in the proof of Lemma 9.20, applying then the continuity principle and using the fact that the mapping (9.24) is one-to-one, after possibly shrinking Ω in a neighborhood of p_1, and shrinking $\varepsilon > 0$, we deduce that for each $u \neq 0$, there exists a holomorphic function $F \in \mathcal{O}\left(\Omega \cup \mathcal{W}_2 \cup \mathcal{W}_u^1\right)$ with $F_{|\Omega \cup \mathcal{W}_2} = f$.

To conclude the proof of Proposition 5.12 in Case **(II)**, it suffices to observe that for every fixed small u with $-\varepsilon << u < 0$, the wedge $\mathcal{W}_u^1$ contains in fact a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$.

The proofs of Proposition 5.12 and of Theorem 3.19 **(i)** are complete now. □

9.27. End of proof of Theorem 1.2'. In order to derive Theorem 1.2' from Theorem 3.19 **(i)**, we now remind the necessity of supplementary arguments about the stability of our constructions under deformation. Coming back to the strategy developped in §3.16, we had a wedge $\mathcal{W}_1$ attached to $M \backslash C_{\rm nr}$. Using a partition of unity, we may introduce a one-parameter $\mathcal{C}^{2,\alpha}$-smooth family of generic submanifolds M^d, $d \in \mathbb{R}$, $d \geq 0$, with $M^0 \equiv M$, with M^d containing $C_{\rm nr}$ and with $M^d \backslash C_{\rm nr}$ contained in $\mathcal{W}_1$. In the proof of Theorem 3.19 **(i)**, thanks to this deformation, the wedge $\mathcal{W}_1$ was replaced by a neighborhood Ω of $M \backslash C_{\rm nr}$ in $\mathbb{C}^n$.

In Sections 4 and 5, we constructed an important semi-local half-wedge $(\mathcal{HW}_\gamma^+)^d$ attached to a one-sided neighborhood of $(M^1)^d$ in M^d along a characteristic segment γ^d of M^d. Now, we make the crucial claim that, after possibly adapting the deformation M^d, we may achieve that the geometric extent of this semi-local half-wedge be uniform as $d > 0$ tends to zero, namely $(\mathcal{HW}_\gamma^+)^d$ tends to a semi-local half-wedge $(\mathcal{HW}_\gamma^+)^0$ attached to a one-sided neighborhood of M^1 in M along γ, as d tends to zero. Indeed, in Section 4 we have constructed a family of analytic discs $(\mathcal{Z}_{t,\chi,\nu:s}(\zeta))^d$ (*cf.* (4.61)) which covers the half-wedge $(\mathcal{HW}_\gamma^+)^d$. Thanks to the stability of Bishop's equation under $\mathcal{C}^{2,\alpha}$-smooth perturbations, the deformed family $(\mathcal{Z}_{t,\chi,\nu:s}(\zeta))^d =: \mathcal{Z}^d_{t,\chi,\nu:s}(\zeta)$ is also of class $\mathcal{C}^{2,\alpha-0}$ with respect to the parameter d. We remind that for every $d > 0$, the family $\mathcal{Z}^d_{t,\chi,\nu:s}(\zeta)$ was in fact constructed by means of a family $\widehat{Z}^d_{r_0,t,\tau,\chi,\nu:s}(\zeta)$ obtained by solving Bishop's equation (4.40), where we now add the parameter d in the function Φ'. In order to construct the semi-local attached half-wedge, we have used the rank property stated in Lemma 4.34. This rank property relied on the possibility of deforming the disc $\widehat{Z}_{r_0,t:s}(\zeta)$ near the point $\widehat{Z}^d_{r_0,t:s}(-1)$ in the open neighborhood $\Phi_s(\Omega) \equiv \Phi_s\left(\mathcal{W}_1\right)$ of $\Phi_s\left(M^d\right)$. As $d > 0$ tends to zero, if M^d tends to M, the size of the neighborhood $\Phi_s\left(\mathcal{W}_1\right)$ shrinks to zero, hence it could seem that the we have no control on the semi-local attached half-wege $(\mathcal{HW}_\gamma^+)^d$ as $d > 0$ tends to zero. Fortunately, since the points $\widehat{Z}^d_{r_0,0:s}(-1)$ in a neighborhood of which we introduce the deformations (4.30) are at a

uniformly positive distance $\delta > 0$ from γ, we may choose the deformation M^d of M to tend to M as d tends to zero only in a small neighborhood of γ, whose size is small in comparison to this distance δ. By smoothness with respect to d of the family $\mathcal{Z}^d_{t,\chi,\nu:s}(\zeta)$, we then deduce that the semi-local half-wedge $(\mathcal{HW}^+_\gamma)^d$ tends to a nontrivial semi-local half-wedge $(\mathcal{HW}^+_\gamma)^0$ as d tends to zero, which proves the claim.

Next, again thanks to the stability of Bishop's equation under perturbation, all the constructions of Sections 5, 6, 7, 8 and 9 above may be achieved to depend $\mathcal{C}^{2,\alpha-0}$-smoothly, hence uniformly, with respect to d. Importantly, we observe that if the deformation M^d is chosen so that M^d tends to M only in a small neighborhood of p_1 of size $<< \varepsilon$, then the shrinking of ε which occurs in Lemma 9.13 may be achieved to be uniform as d tends to zero, because the part $A_{x,v,u:c}(\partial^3\Delta)$ stays in a uniform compact subset of Ω, as d tends to zero. At the end of the proof of Proposition 5.12, we then obtain univalent holomorphic extension to a local wedge $\mathcal{W}^d_{p_1}$ of edge M^d or to a neighborhood $\omega^d_{p_1}$ of M^d in $\mathbb{C}^n$, and they tend smoothly to a wedge $\mathcal{W}^0_{p_1}$ of edge M at p_1 or to a neighborhood ω_{p_1} of p_1 in $\mathbb{C}^n$.

The proof of Theorem 1.2' is complete. $\square$

§10. THREE PROOFS OF THEOREM 1.4

10.1. Preliminary. Theorem 1.4 may be established by means of the processus of minimalization of generic submanifods developed by B. Jöricke in [J2], as is done effectively in [J4] in the hypersurface case and then in [P2] in arbitrary codimension. In this section, we shall suggest three more different proofs of Theorem 1.4. As explained in Section 3, it suffices to treat $\mathcal{W}$-removability, and essentially to prove Proposition 3.22. The first proof appears already in [M2] and also in [P1]. The second proof consists in repeating some of the constructions of the previous sections, using the fact that M^1 is of positive CR dimension in order to simplify substantially the reasonings. The third proof consists of a slicing argument showing that *Theorem 1.4 is in fact a logical consequence of Theorem 1.2'*. In fact, because these three proofs are already written elsewhere or very close to the constructions developed in the previous sections, we shall only provide summaries here.

10.2. Normal deformation of analytic discs attached to M^1. Firstly, in the situation of Proposition 3.22, because M^1 is of positive CR dimension, we can construct a small analytic disc $A(\zeta)$ attached to M^1 which satisfies $A(1) = p_1 \in M^1$ and $A(\partial\Delta\backslash\{1\}) \subset (H^1)^+$. As in [M2], [MP1], using normal deformations of A near $A(-1)$, we may include A in a $\mathcal{C}^{2,\alpha-0}$-smooth parametrized family $A_v(\zeta)$ of analytic discs attached to M^1, where $v \in \mathbb{R}^{d+1}$ is small, so that the rank at $v = 0$ of the mapping $v \longmapsto -\frac{\partial A_v}{\partial \rho}(1) \in T_{p_1}\mathbb{C}^n \bmod T_{p_1}M^1 \cong \mathbb{R}^{d+1}$ is maximal equal to $(d+1)$, the codimension of M^1 in $\mathbb{C}^n$. For this, we use a deformation lemma which is essentially due to A. Tumanov [Tu3], which appears as Lemma 2.7 in [MP1] and which was already used above (with a supplementary parameter s) in Lemma 4.34. Then we add a "translation" parameter $x \in \mathbb{R}^{2m+d-1}$, getting a family of analytic discs $A_{x,v}(\zeta)$ with $A_{x,v}(\partial\Delta) \subset M^1$ so that the rank at $x = 0$ of the mapping $x \longmapsto A_{x,0}(1) \in M^1$ is maximal equal to $(2m+d-1)$, the dimension of M^1. Finally, we introduce some "translations" M^1_u of M^1 in M, where $u \in \mathbb{R}$, and we obtain a family $A_{x,v,u}(\zeta)$ of analytic discs attached to M^1_u. For $u \neq 0$, since the discs $A_{x,v,u}(\zeta)$ are attached to M^1_u, their boundaries are contained in $M\backslash C$. Applying the approximation Lemma 4.8, we deduce that for every $u \neq 0$, all holomorphic

functions in the open set Ω of Proposition 3.22 (which contains $M\backslash C$) extend holomorphically to the local wedge $\mathcal{W}_u := \{A_{x,v,u}(\rho) : \ |x| < \varepsilon, \ |v| < \varepsilon, \ 1-\varepsilon < \rho < 1\}$ of edge M_u^1. To control the univalence of the holomorphic extension, it suffices to shrink Ω a little bit in a neighborhood of p_1. To conclude the proof, one observes as in [J4], [CS] that the union $\bigcup_{u\neq 0} \mathcal{W}_u^1$ contains a local wedge of edge M at p_1. This processus is sometimes called "sweeping out by wedges". We notice that this proof is geometrically much simpler than the proof of Theorem 1.2' achieved in the previous sections.

10.3. Half-attached analytic discs. Secondly, we may generalize our constructions achieved in the previous sections from the CR dimension $m = 1$ case to the CR dimension $m \geq 2$ case, as follows. Let M, M^1, p_1, H^1 and C be as in Proposition 3.22. The fact that M^1 is of positive CR dimension will provide substantial geometric simplifications for essentially two reasons. Indeed, as the hypersruface $H^1 \subset M^1$ is generic and at least of real dimension n, there exists a (in fact infinitely many) maximally real submanifold K^1 passing through p_1 which is contained in H^1. Then

- **(i)** Applying the considerations of Section 7, we may construct families of small analytic discs $A_{x,v}(\zeta)$ which are half-attached to K^1 and which cover a local wedge of edge K^1 at p_1, when the translation parameter x and the rotation parameter v vary. We notice that this would be impossible in the case $m = 1$, because in this case H^1 is of real dimension $(n-1)$, hence does *not* contain any maximally real submanifold.
- **(ii)** We can even prescribe the direction $\frac{\partial A_{0,0}}{\partial \theta}(1)$ as an arbitrary given nonzero vector $v_1 \in T_{p_1}K^1$. Again, this would be impossible in the CR dimension $m = 1$ case.

Let $\mathcal{HW}_1^+$ be a local half-wedge of edge $(M^1)^+$ at p_1, whose construction is suggested in §4.64. Since K^1 is generic, we may choose a nonzero vector $v_1 \in T_{p_1}K^1$ with the property that $\mathcal{HW}_1^+$ is directed by Jv_1. Generalizing Lemma 8.3, we see that $A_{x,v}\left(\overline{\Delta}\backslash\partial^+\Delta\right)$ is contained in $\mathcal{HW}_1^+$. We notice that in the CR dimension $m = 1$ case, H^1 is *not* generic, and we remember that the choice of a special point p_1 to be removed locally and the choice of a supporting hypersurface $H^1 \subset M^1$ was much more subtle, because we had to insure that there exists a vector $v_1 \in T_{p_1}H^1$ such that $\mathcal{HW}_1^+$ is directed by Jv_1.

Next, we can translate K^1 in M^1 by means of a small parameter $t \in \mathbb{R}^{d-1}$ and then M^1 in M by means of a small parameter $u \in \mathbb{R}$. By stability of Bishop's equation, we get a family of analytic discs $A_{x,v,t,u}(\zeta)$ half-attached to the translation $K_{t,u}^1$ of K^1. Nine properties analogous to properties $\mathbf{(1_1)}$ to $\mathbf{(9_1)}$ of Lemmas 7.12 and 8.3 are then satisfied and we conclude the proofs of Proposition 3.22 and of Theorem 1.4 in essentially the same way as in Section 9 above. We shall not write down all the details.

We notice that this second strategy of proof is much more complicated than the first (known) proof summarized in §10.2 just above.

10.4. Slicing argument: reduction of Theorem 1.4 to Theorem 1.2'. Thirdly, we may provide a new proof of the central Theorem 1.4 valid in CR dimension $m \geq 2$, in order to illustrate how our results in CR dimension 1 can be applied to the general case via slicing techniques. We shall see that Theorem 1.4 is a logical consequence of Theorem 1.2'.

Let M, M^1 and C be as in Theorem 1.4. To begin with, we shall treat the three notions of removability (CR-, L^{p}- and $\mathcal{W}$-) commonly. However, we remind that CR-removability is immediately reduced to $\mathcal{W}$-removability thanks to Lemma 3.5, hence it suffices to consider only L^{p}- and $\mathcal{W}$-removability.

Arguing by contradiction, we see as in the previous parts of this paper that we lose essentially nothing if we consider C to be the minimal nonremovable subset. Also, we

may assume holomorphic extension to a wedge attached to $M\backslash C$. As explained in §3.16 it is enough to remove one single point of C.

As in §3.16 (*see* especially the statement of Proposition 3.22), we can show that there is a $\mathcal{C}^{2,\alpha}$-smooth hypersurface H of M which is generic in $\mathbb{C}^n$, transversal to M^1, and which has the following property: H contains some point $p_1 \in C$, and we can choose a small neighborhood of H in M in which H has two connected open sides H^+ and H^- such that C is contained in $H^- \cup \{p_1\}$ locally in a neighborhood of p_1. Notice that the hypersurface H^1 constructed in Lemma 3.21 is simply the intersection of M^1 with such a hypersurface H.

Next, in a neighborhood of p_1, we construct a local foliation of M by generic submanifolds of CR dimension 1 (hence of dimension $n+1$) as follows. We notice that $\dim_{\mathbb{R}} H = 2m + d - 1$. We first choose a local $\mathcal{C}^{2,\alpha}$-smooth foliation of H by generic submanifolds of CR dimension 1 (hence of real dimension $n+1$) which we denote by $\widetilde{M}_s$, where the transversal parameters $s = (s_1, \dots, s_{m-2}) \in \mathbb{I}_{m-2}(\rho_1)$ belong to a cube in $\mathbb{R}^{m-2}$ of radius some $\rho_1 > 0$. Afterwards, we extend this foliation to a local $\mathcal{C}^{2,\alpha}$-smooth foliation $\widetilde{M}_{s,t}$ of a neighborhood of p_1 in M, where $s \in \mathbb{I}_{m-2}(\rho_1)$, where $t \in \mathbb{I}(\rho_1)$ and where $\widetilde{M}_{s,0} \equiv \widetilde{M}_s$; if $m = 2$, we notice that the parameter s disappears and that we have only one real parameter t. Also, we can assume that $\widetilde{M}_{s,t}$ is contained in H^+ if and only if $t > 0$.

By genericity, the submanifolds $\widetilde{M}_{s,t}$ may be chosen in addition to be transversal to the one-codimensional submanifold $M^1 \subset M$ containing the singularity C with of course $p_1 \in \widetilde{M}_{0,0}$. Then for all parameters s and t the intersections $\widetilde{M}^1_{s,t} := M^1 \cap \widetilde{M}_{s,t}$ are maximally real submanifolds of $\mathbb{C}^n$. Considering $\widetilde{M}^1_{s,t}$ as a maximally real one-codimensional submanifold of $\widetilde{M}_{s,t}$, a characteristic foliation is induced on each $\widetilde{M}^1_{s,t}$. After contraction around p_1 we can assume that these characteristic foliations all have trivial topology: their leaves are the level sets of an $\mathbb{R}^{m-1}$-valued submersion. As the intersection $\widetilde{M}^1_{0,0} \cap C$ is the singleton $\{p_1\}$ (by construction of H), there exists $\varepsilon > 0$ such that for all s with $|s| < \varepsilon$ and all t with $|t| < \varepsilon$, the closed subsets $C_{s,t} := \widetilde{M}^1_{s,t} \cap C$ are compact in $\widetilde{M}^1_{s,t}$. Notice that $C_{s,t}$ is even empty if $t > 0$, because C is contained in $H^- \cup \{p_1\}$. The following simple fact shows that $C_{s,t}$ satisfies the nontransversality condition $\mathcal{F}^c_{\widetilde{M}^1_{s,t}}\{C_{s,t}\}$ of Theorem 1.2', for all s with $|s| < \varepsilon$ and all t with $-\varepsilon < t \leq 0$.

Lemma 10.5. *Let $\mathcal{F}_{\mathcal{M}}$ be a $\mathcal{C}^{1,\alpha}$-smooth foliation by curves on some m-dimensional $\mathcal{C}^{2,\alpha}$-smooth real manifold $\mathcal{M}$ defined by a surjective $\mathcal{C}^{1,\alpha}$-smooth submersion $F : \mathcal{M} \to \mathbb{I}_{m-1}(\rho_1)$. Then every compact set $\mathcal{C} \subset \mathcal{M}$ satisfies the nontransversality condition $\mathcal{F}_{\mathcal{M}}\{\mathcal{C}\}$.*

Proof. Let $\mathcal{C}'$ be an arbitrary compact subset of $\mathcal{C}$. As $\mathcal{C}'$ is compact, there exists the smallest $\rho_2 < \rho_1$ with $\mathcal{C}' \subset F^{-1}(\overline{\mathbb{I}_{m-1}(\rho_2)})$. The semi-local projection $\pi_{\mathcal{F}_{\mathcal{M}}}$ along the leaves of $\mathcal{F}$ may of course be identified with F. Thus, $\pi_{\mathcal{F}_{\mathcal{M}}}(\mathcal{C}')$ is contained in $\overline{\mathbb{I}_{m-1}(\rho_2)}$ and meets the boundary $\partial \mathbb{I}_{m-1}(\rho_2)$ of the cube $\mathbb{I}_{m-1}(\rho_2)$. Also, by compactness, the set $\mathcal{C}'$ cannot contain a fiber of F in the whole. This completes the proof. □

We can now show that Theorem 1.4 is a logical consequence of Theorem 1.2'. In fact, we cannot insure that the generic submanifolds $\widetilde{M}_{s,t}$ of CR dimension 1 defined above are all globally minimal, hence it seems that Theorem 1.2' itself does not apply. However, we notice that the wedge attached to $M\backslash C$ restricts to a wedge attached to $\widetilde{M}_{s,t}\backslash C_{s,t}$, for all s and t. Hence, we can observe that everything that was needed in the proof

of Theorem 1.2' was the existence of a wedge attached to $M\backslash C$ to which holomorphic extension is already assumed. One can even formulate a slightly more general version of Theorem 1.2', where the global minimality assumption is replaced by the assumption of holomorphic extension to a wedge attached to $M\backslash C$. Of course, with this more general assumption, M may consists of several CR orbits, but thanks to Lemma 3.5 about stability of CR orbits, one may check that the proof of the main Theorem 3.19 **(i)** and of the main Proposition 5.12 remain unchanged, in the case where holomorphic extension is assumed in a wedgelike domain over $M\backslash C$ and not only in wedgelike domains attached to the CR orbits of $M\backslash C$.

Thus this slight generalization of Theorem 1.2' together with the observation made in Lemma 10.5 just above yield that for all s with $|s| < \varepsilon$ and for all t with $-\varepsilon < t \leq 0$, the closed subset $C_{s,t}$ is $\mathcal{W}$-removable in the generic submanifold $\widetilde{M}_{s,t}$. We deduce that for every (s,t) with $|s| < \varepsilon$ and $-\varepsilon < t \leq 0$, we get holomorphic extension from the given restricted wedge attached to $\widetilde{M}_{s,t}\backslash C_{s,t}$ into an open wedge $\widetilde{\mathcal{W}}_{s,t}$ attached $\widetilde{M}_{s,t}$. Notice that this does not immediately achieves the proof, since the direction of the $\widetilde{\mathcal{W}}_{s,t}$ need not depend continuously on (s,t). In fact, the proof of the slight generalization of Theorem 1.2' contains arguments (for example the localization near a very special point) which do not depend nicely on external parameters. Hence the attached wedges $\widetilde{\mathcal{W}}_{s,t}$ may well be completely unrelated.

To overcome this difficulty we proceed in the following way, already argued in [M2], [MP1] (Lemma 2.7) in slightly different contexts. We first construct a regular family $A_{x,v}(\zeta)$ of analytic discs attached to $\widetilde{M}_{0,0} \cup \Omega$ whose size is small in comparison to the basis of the wedge $\widetilde{\mathcal{W}}_{0,0}$ and which sweep out a local wedge $\mathcal{W}(A_{x,v})$ of edge $\widetilde{M}_{0,0}$ at p_1. Here, the parameter $x \in \mathbb{R}^{n+1}$ corresponds to translations in $\widetilde{M}_{0,0}$ and $v \in \mathbb{R}^{n-2}$ to normal deformations in a neighborhood of the point $A_{0,0}(-1) \in \Omega$. Deforming this family thanks to the flexibility of Bishop's equation, we construct a family $A_{x,v,s,t}(\zeta)$ attached to $\widetilde{M}_{s,t} \cup \Omega$, still sweeping out a local wedge $\mathcal{W}(A_{x,v,s,t})$ of edge $\widetilde{M}_{s,t}$. This family is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all parameters. Using the $\mathcal{W}$-removability of $C_{s,t}$, we can introduce for every (s,t) a one-parameter deformation $\widetilde{M}_{s,t}^{d}$ of $\widetilde{M}_{s,t}$ which is contained in the attached wedge $\widetilde{W}_{s,t}$ whenever $d > 0$ and which coincides with $\widetilde{M}_{s,t}$ when $d = 0$. Thanks to the flexibility of Bishop's equation with parameters, we get a deformed family $A_{x,v,s,t:d}(\zeta)$ of analytic discs. Since the wedges $\widetilde{W}_{s,t}$ are *a priori* unrelated, we loose the smoothness with respect to all variables, including d. Fortunately, by an application of the continuity principle, for every $d > 0$, we deduce holomorphic extension to the wedge generated by the family $A_{x,v,s,t:d}(\zeta)$. If we let d tend to zero, fixing (s,t), we obtain univalent holomorphic extension to the wedge generated by the family $A_{x,v,s,t}(\zeta)$. Finally, as (s,t) varies, the wedges $\mathcal{W}(A_{x,v,s,t})$ varies smoothly and covers a local wedge $\mathcal{W}$ of edge M at p_1. By the continuity principle, we may verify that we obtain univalent holomorphic extension to $\mathcal{W}$.

Secondly, we explain how L^{p}-removability of the point $p_1 \in C$ in M follows logically from the L^{p}-removability of every $C_{s,t}$ in $\widetilde{M}_{s,t}$. The main argument relies on the following simple but useful fact: if M is a generic CR submanifold of $\mathbb{C}^n$ and if $N \subset M$ is a lower dimensional submanifold which is itself a generic CR submanifold of $\mathbb{C}^n$ of positive CR dimension, then differentiable CR functions on M obviously restrict to CR functions on N. More generally, a foliated version of this observation with lower regularity assumptions is as follows.

Lemma 10.6. *Let $M \subset \mathbb{C}^n$ be a generic submanifold of class $\mathcal{C}^{2,\alpha}$ and of CR dimension $m \geq 2$.*

- **(a)** *If $p_1 \in M$ and if M carries a local $\mathcal{C}^{2,\alpha}$-smooth foliation by a family N_u of generic submanifolds of CR dimension 1 where $u \in \mathbb{R}^{m-1}$ is a small parameter and where $p_1 \in N_0$, then for every CR function $f \in L^{\mathsf{p}}_{loc}(M)$, $\mathsf{p} \geq 1$, and for almost every $u \in \mathbb{R}^{m-1}$, its restriction $f|_{N_u}$ is an $L^{\mathsf{p}}_{loc}(N_u)$ function which is CR on N_u.*
- **(b)** *Conversely, if $p_1 \in M$ and if M carries m local $\mathcal{C}^{2,\alpha}$-smooth foliations by families $N^j_{u_j}$, $j = 1, \dots, m$, $u_j \in \mathbb{R}^{m-1}$, of generic submanifolds of CR dimension 1 satisfying $p_1 \in N^j_0$ for $j = 1, \dots, m$ and*

$$T_{p_1} N^1_0 + \cdots + T_{p_1} N^m_0 = T_{p_1} M, \tag{10.7}$$

then a function $f \in L^{\mathsf{p}}_{loc}(M)$ is CR in a neighborhood of p_1 if and only if for all $j = 1, \dots, m$ and for almost every $u_j \in \mathbb{R}^{m-1}$, its restriction $f|_{N^j_{u_j}}$ is CR on $N^j_{u_j}$.

Proof. Of course, property **(a)** only makes sense for an everywhere defined representative of f and the nullset of excluded parameters t depends on the choice of the representative of f.

To establish **(a)**, we choose a small box-neighborhood $U \cong \mathbb{I}_{m-1}(\rho_1) \times \widetilde{N}$, where $\mathbb{I}_{m-1}(\rho_1)$ is a cube of some positive radius $\rho_1 > 0$ in $\mathbb{R}^{m-1}$, such that every plaque $\{\upsilon\} \times \widetilde{N}$ is an open subset of some leaf N_u. By the L^{p} version of the approximation theorem (*cf.* [J5], [P1], [MP1]), the restriction $f|_U$ is the limit in the L^{p} norm of the restrictions of holomorphic polynomials $(P_\nu)_{\nu \in \mathbb{N}}$. Thanks to Fubini's theorem, we deduce that for almost every $\upsilon \in \mathbb{I}_k(\rho_1)$, the restriction $P_\nu|_{\{\upsilon\} \times \widetilde{N}}$ converges in L^{p} norm to $f|_{\{\upsilon\} \times \widetilde{N}}$. Hence for such parameters υ, the restriction $f|_{\{\upsilon\} \times \widetilde{N}}$ is CR, which completes the proof of **(a)**.

To establish **(b)**, we observe first that the "only if" part is a direct consequence of **(a)**. To prove the "if" part, we may introduce for every $j = 1, \dots, m$ and for every $u_j \in \mathbb{R}^{m-1}$ a $(0,1)$ vector field $\overline{L}^j_{u_j}$ tangent to $N^j_{u_j}$ and $\mathcal{C}^{1,\alpha}$-smoothly parameterized by u_j. The geometric assumption (10.7) entails that the m vector fields $\overline{L}^1_{u_1}, \dots, \overline{L}^m_{u_m}$ generate the CR bundle $T^{0,1}M$ in a neighborhood of p_1. By assumption, the L^{p}_{loc} function f is annihilated in the distributional sense by these m vector field, hence it is CR. This completes the proof of **(b)**. □

We can now prove that the L^{p}-removability of C in M follows from an application of Theorem 1.2'. Let a function $f \in L^{\mathsf{p}}_{loc}(M)$ which is CR on $M \backslash C$. Coming back to the construction of the submanifolds $\widetilde{M}_{s,t}$ achieved in the paragraphs before Lemma 10.5, it is clear that for almost every $(s,t) \in \mathbb{R}^{m-1}$, the restriction $f|_{\widetilde{M}_{s,t}}$ is L^{p}_{loc}-integrable. More generally, proceeding as in the paragraph before Lemma 10.5, we may construct m such families $\widetilde{M}_{j;s_j,t_j}$ for $j = 1, \dots, m$ with $p_1 \in \widetilde{M}_{j;0,0}$ and

$$T_{p_1} \widetilde{M}_{1;s_1,t_1} + \cdots + T_{p_1} \widetilde{M}_{m;s_m,t_m} = T_{p_1} M, \tag{10.8}$$

without changing the conclusion that the corresponding closed subsets $C^j_{s_j,t_j}$ are L^{p}-removable in $\widetilde{M}_{j;s_j,t_j}$ for $j = 1, \dots, m$. Applying Lemma 10.7 just above, we finally deduce that f is CR on M, as desired.

This completes the description of the reduction of Theorem 1.4 to Theorem 1.2' via a slicing argument.

10.9. Version of Theorem 1.4 in an open subset of $\mathbb{C}^n$. To conclude this section, we remind that in the end of the proof of Proposition 5.12 in Cases **($\mathbf{I_1}$)** and **($\mathbf{I_2}$)**, we came down to the removability of a proper closed subset $\mathcal{C}_{p_1}$ of a one-codimensional submanifold $\mathcal{M}_{p_1}$ of an open subset of $\mathbb{C}^n$ ($n \geq 2$), namely the wedge $\mathcal{W}_{p_1}$ (remind FIGURE 18 above), which amounts exactly to prove Theorem 1.4 in the case where the generic submanifold M is replaced by an open subset of $\mathbb{C}^n$. We may formulate this result as the following lemma. To our knowledge, its first known proof is given in [J4]. Here, we provide a slightly different proof, using half-attached analytic discs.

Lemma 10.10. *Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain, let $\mathcal{M}^1 \subset \mathcal{D}$ be a connected $\mathcal{C}^{2,\beta}$-smooth hypersurface with $0 < \beta < 1$ and let $\mathcal{C}$ be a proper closed subset of $\mathcal{M}^1$ which does not contain any CR orbit of $\mathcal{M}^1$. Then for every holomorphic function $f \in \mathcal{O}(\mathcal{D}\backslash\mathcal{C})$, there exists a holomorphic function $F \in \mathcal{O}(\mathcal{D})$ such that $F|_{\mathcal{D}\backslash\mathcal{C}} = f$.*

Proof. We summarize the proof, which anyway is very similar to the proof of Proposition 3.22 delineated in §10.3 above. Reasoning by contradiction and constructing a supporting hypersurface, we come down to the local removability of a single point p_1 in a geometric situation analogous to the one described in Proposition 3.22, with M replaced by the domain $\mathcal{D}$, with M^1 replaced by $\mathcal{M}^1$ and with a generic $\mathcal{C}^{2,\beta}$-smooth submanifold $\mathcal{H}^1 \subset \mathcal{M}^1$ such that, locally in a neighborhood of p_1, we have $\mathcal{C} \subset (\mathcal{H}^1)^- \cup \{p_1\}$, where we use the same notation $\mathcal{C}$ for the smallest non-removable subset of the original $\mathcal{C}$.

Notice that $\mathcal{H}^1$ is of codimension 2. Let $\mathcal{K}^1 \subset \mathcal{H}^1$ be a maximally real submanifold passing through p_1. We may translate $\mathcal{K}^1$ in $\mathcal{M}^1$ by means of a small parameter $t \in \mathbb{R}^{n-1}$ and then $\mathcal{M}^1$ in $\mathcal{D}$ by means of a small parameter $u \in \mathbb{R}$. Following Section 7, we then construct a small family of analytic discs $\mathcal{A}^1_{x,v,t,u}(\zeta)$ half-attached to the "translations" $\mathcal{K}^1_{t,u}$. Nine properties analogous to properties **($\mathbf{1_1}$)** to **($\mathbf{9_1}$)** of Lemmas 7.12 and 8.3 are then satisfied and we conclude the proof in essentially the same way as in Section 9 above. □

§11 $\mathcal{W}$-REMOVABILITY IMPLIES L^{p}-REMOVABILITY

11.1. Preliminary. This section is devoted to prove Lemma 3.15 about L^{p}-removability of the proper closed subset $C \subset M^1$, granted it is $\mathcal{W}$-removable. More generally, we shall establish the L^{p}-removability of certain proper closed subsets Φ of M that are nullsets with respect to the Lebesgue measure of M.

As a preliminary, we remind that if M' is a globally minimal $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 1$ and of codimension $d = n - m \geq 1$, there exists a wedge $\mathcal{W}'$ attached to M' constructed by means of analytic discs successively glued to M' and to conelike submanifolds attached to M' consisting of parametrized families of pieces of analytic discs. By means of the approximation theorem of [BT], one deduces classically that continuous CR functions on M' extend holomorphically to $\mathcal{W}'$, and continuously to $M' \cup \mathcal{W}'$.

For the holomorphic extension of the L^{p}_{loc} CR functions to a wedge attached to M', some supplementary routine, though not obvious, work has to be achieved. Firstly, using a convolution with Gauss' kernel as in [BT], one shows that on a $\mathcal{C}^2$-smooth generic submanifold M' of $\mathbb{C}^n$, every L^{p}_{loc} CR function on M' is locally the limit, in the L^{p} norm, of a sequence of polynomials (*see* Lemma 3.3 in [J5]). In the case where M' is a hypersurface, studied in [J5], the wedge $\mathcal{W}'$ is in fact a one-sided neighborhood attached to

M', which we will denote by $\mathcal{S}'$. The theory of Hardy spaces on the unit disc transfers to parameterized families of small analytic discs glued to M' which cover local one-sided neighborhoods of a hypersurface, provided the boundaries of these discs foliate an open subset of M'. Using in an essential way L. Carleson's imbedding theorem, B. Jöricke established in [J5] that every L^{p}_{loc} CR function on a globally minimal $\mathcal{C}^2$-smooth hypersurface M' extends holomorphically in the Hardy space $H^{\mathsf{p}}(\mathcal{S}')$ of holomorphic functions defined in the one-sided neighborhood $\mathcal{S}'$, with L^{p} boundary values on the hypersurface M'. In his thesis [P1], the second author of the present paper has built the theory in higher codimension, introducing the Hardy space $H^{\mathsf{p}}(\mathcal{W}')$ of functions holomorphic in the wedge $\mathcal{W}'$ attached to M', with L^{p} boundary values on the edge M'.

At present, these background statements about holomorphic extendability of L^{p}_{loc} CR functions on globally minimal generic submanifolds may be reproved in a more elegant way than by going through the rather complicated technology dispersed in the articles [Tu1], [Tu2], [M1], [J2], thanks to a simplification of the wedge extendability theorem obtained recently by the second author of this paper, which treats in an unified way local and global minimality. We refer the reader to the work in preparation [P3] for a substantial cleaning of the theory.

11.2. L^{p}**-removability of nullsets.** Let us say that a subset Φ of a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold is *stably $\mathcal{W}$-removable* if it is $\mathcal{W}$-removable on every compactly supported sufficiently small $\mathcal{C}^{2,\alpha}$-smooth deformation M^d of M leaving Φ fixed. In the situations of Theorems 1.2' and 1.4, the assumptions of Lemma 11.3 just below are satisfied with $\Phi = C$, taking account of the fact that we have already established the $\mathcal{W}$-removability of C and that for logical reasons only, the closed set C in the statements of Theorems 1.2' and 1.4 is obviously stably removable.

Lemma 11.3. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of CR dimension $m \geq 1$ and of codimension $d = n - m \geq 1$, hence of dimension $(2m+d)$, let $\Phi \subset M$ be a nonempty proper closed subset whose $(2m+d)$-dimensional Hausdorff measure is equal to zero. Assume that $M \backslash \Phi$ is globally minimal and let $\mathcal{W}$ be a wedge attached to $M \backslash \Phi$ such that every function in $L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$ extends holomorphically as a function in the Hardy space $H^{\mathsf{p}}(\mathcal{W})$. If Φ is stably $\mathcal{W}$-removable, then Φ is L^{p}-removable.*

Before giving the proof, let us summarize intuitively the reason why this strong L^{p}-removability result Lemma 11.3 holds. Indeed, let $f \in L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$. As soon as wedge extension over points of Φ is known, thanks to the fact that we can deform M over Φ in the wedgelike domain, thus erasing the singularity Φ, we get a L^{p}_{loc} CR function f^d on the deformed manifold M^d, without singularities anymore, and in addittion, we can let the deformation M^d tend to M with a uniform with L^{p} control of the extension f^d, which therefore tends to a CR extension of f through Φ.

Proof. First of all, we remind that for every p with $1 \leq \mathsf{p} \leq \infty$, the space $L^{\mathsf{p}}_{loc}(M)$ is contained in $L^1_{loc}(M)$. We claim that it follows that Φ is L^{p}-removable for every p with $1 \leq \mathsf{p} \leq \infty$ if and only if Φ is L^1-removable. Indeed, suppose that Φ is L^1-removable, namely for every function $f \in L^1_{loc}(M) \cap \mathrm{CR}(M \backslash \Phi)$, and every $\mathcal{C}^1$-smooth $(n, m-1)$-form with compact support, we have $\int_M f \cdot \overline{\partial} \psi = 0$. In particular, since L^{p}_{loc} is contained in L^1_{loc} by Hölder's inequality, this property holds for every function $g \in L^{\mathsf{p}}_{loc}(M) \cap CR(M \backslash \Phi)$, hence Φ is L^{p}-removable, as claimed. Consequently, it suffices to show that $\mathcal{W}$-removability implies L^1-removability.

Let $f \in L^1_{loc}(M\backslash\Phi) \cap L^1(M)$ be an arbitrary function. The goal is to show that f is in fact CR on Φ. Of course, it suffices to show that f is CR locally at every point of Φ. So, we fix an arbitrary point $q \in \Phi$. If ψ is an arbitrary $(n, m-1)$-form of class $\mathcal{C}^1$ supported in a sufficiently small neighborhood of q, we have to prove that $\int_M f \cdot \overline{\partial}\psi = 0$.

We may also fix a small open polydisc $\mathcal{V}_q$ centered at q. We shall first argue that we can assume that the L^1_{loc} function f is holomorphic in a neighborhood of $(M\backslash\Phi)\cap\mathcal{V}_q$ in $\mathbb{C}^n$. Indeed, since $M\backslash\Phi$ is globally minimal, there exists a wedge $\mathcal{W}$ attached to $M\backslash\Phi$ such that every L^1_{loc} CR function on $M\backslash\Phi$, and in particular f, extends holomorphically as a function which belongs to the Hardy space $H^1(\mathcal{W})$. By slightly deforming $(M\backslash\Phi)\cap\mathcal{V}_q$ into $\mathcal{W}$ along Bishop discs glued to $M\backslash\Phi$, keeping Φ fixed, using the theory of Hardy spaces in wedges developed in [P1], we may obtain the following deformation result with L^1 control, a statement which is a particular case of Proposition 1.16 in [MP1].

Proposition 11.4. *For every $\varepsilon > 0$, every $\beta < \alpha$, there exists a small $\mathcal{C}^{2,\beta}$-smooth deformation M^d of M with support contained in $\overline{\mathcal{V}_q}$ and there exists a function $f^d \in L^1_{loc}\left(M^d\right) \cap CR\left(M^d\backslash\Phi\right)$, such that*

- **(1)** $M^d \cap \mathcal{V}_q \supset \Phi \cap \mathcal{V}_q \ni q$.
- **(2)** $\left(M^d\backslash\Phi\right) \cap \mathcal{V}_q \subset \mathcal{W} \cap \mathcal{V}_q$.
- **(3)** *f^d is holomorphic in the neighborhood $\mathcal{W}\cap\mathcal{V}_q$ of $(M^d\backslash\Phi)\cap\mathcal{V}_q$ in $\mathbb{C}^n$.*
- **(4)** *$M \cap \mathcal{V}_q$ and $M^d \cap \mathcal{V}_q$ are graphed over the same $(2m+d)$ linear real subspace and $\left|\left|M^d \cap \mathcal{V}_q - M \cap \mathcal{V}_q\right|\right|_{\mathcal{C}^{2,\beta}} \leq \varepsilon$.*
- **(5)** *The volume forms of $M \cap \mathcal{V}_q$ and of $M^d \cap \mathcal{V}_q$ may be identified and $\left|f - f^d\right|_{L^1(M\cap\mathcal{V}_q)} \leq \varepsilon$.*

Since it will suffice to have a control of the deformation M^d only in $\mathcal{C}^2$ norm, we shall replace $\mathcal{C}^{2,\beta}$ and $\mathcal{C}^{1,\beta}$ by $\mathcal{C}^2$ and $\mathcal{C}^1$ in the sequel.

Let us be more explicit about conditions **(4)** and **(5)**. Without loss of generality, we can assume that in coordinates $(z, w) = (x+iy, u+iv) \in \mathbb{C}^m \times \mathbb{C}^d$ centered at q, we have $T_qM = \{v = 0\}$, hence the generic submanifolds M and M^d are represented locally by vectorial equations $v = \varphi(x, y, u)$ and $v = \varphi^d(x, y, u)$, where φ and φ^d are defined in the real cube $\mathbb{I}_{2m+d}(2\rho_1)$, for some small $\rho_1 > 0$ and that $\mathcal{V}_q$ is the polydisc $\Delta_n(\rho_1)$ of radius ρ_1. Then condition **(4)** simply means that $||\varphi^d - \varphi||_{\mathcal{C}^2(\mathbb{I}_{2m+d}(\rho_1))} \leq \varepsilon$ and condition **(5)** is clear if we choose $dxdydu$ as the volume form on M and on M^d.

Suppose that for every $\varepsilon > 0$ and for every deformation M^d, we can show that the function L^1_{loc} function f^d on M^d is in fact CR over $M^d \cap \Delta_n(\rho_1)$. Then we claim that f is CR in a neighborhood of q.

Indeed, to begin with, let us denote by $\overline{L}_1, \ldots, \overline{L}_m$ a basis of $(0,1)$ vector fields tangent to M, having coefficients depending on the first order derivatives of φ. More precisely, in slightly abusive matrix notation, we can choose the basis $\overline{L} := \frac{\partial}{\partial \bar{z}} + 2(i - \varphi_u)^{-1}\varphi_{\bar{z}}\frac{\partial}{\partial \bar{w}}$. Let us denote this basis vectorially by $\overline{L} = \frac{\partial}{\partial \bar{z}} + A\,\frac{\partial}{\partial \bar{w}}$. To compute the formal adjoint of $\overline{L}$ with respect to the local Lebesgue measure $dxdydu$ on M, we choose two $\mathcal{C}^1$-smooth functions ψ, χ of (x, y, u) with compact support in $\mathbb{I}_{2m+d}(\rho_1)$. Then the integration by part $\int \overline{L}(\psi)\cdot\chi\cdot dxdydu = \int \psi \cdot {}^T\overline{L}(\chi)\cdot dxdydu$ yields the explicit expression ${}^T\overline{L}(\chi) := -\overline{L}(\chi) - A_{\bar{w}}\cdot\chi$ of the formal adjoint of $\overline{L}$.

It follows immediately that if we denote by ${}^T(\overline{L}^d)$ the formal adjoint of the basis of CR vector fields tangent to M^d, then we have an estimate of the form $||{}^T(\overline{L}^d) - {}^T(\overline{L})||_{\mathcal{C}^1} \leq \mathrm{C}\cdot\varepsilon$, for some constant $C > 0$. Recall that f^d is assumed to be CR in $M^d \cap \Delta_n(\rho_1)$. Equivalently, we have $\int f^d \cdot {}^T(\overline{L}^d)(\psi)\cdot dxdydu = 0$ for every $\mathcal{C}^1$-smooth function ψ

with compact support in the cube $\mathbb{I}_{2m+d}(\rho_1)$. Then we deduce that (some explanation follows)

(11.5)

$$\begin{cases} \left|\int f\cdot {}^T\overline{L}(\psi)\cdot dxdydu\right| = \left|\int \left[f\cdot {}^T\overline{L}(\psi) - f^d\cdot {}^T(\overline{L}^d)(\psi)\right]\cdot dxdydu\right| \\ \leq \left|\int \left[f\cdot {}^T\overline{L}(\psi) - f\cdot {}^T(\overline{L}^d)(\psi) + f\cdot {}^T(\overline{L}^d)(\psi) - f^d\cdot {}^T(\overline{L}^d)(\psi)\right]\cdot dxdydu\right| \\ \leq C_1(\psi)\cdot\varepsilon\cdot\int_{\mathbb{I}_{2m+d}(\rho_1)}|f|\cdot dxdydu + C_2(\psi)\cdot\int_{\mathbb{I}_{2m+d}(\rho_1)}|f-f^d|\cdot dxdydu \\ \leq C(\psi,f,\rho_1)\cdot\varepsilon, \end{cases}$$

taking account of property **(5)** of Proposition 11.4 for the passage from the third to the fourth line, where $C(\psi, f, \rho_1)$ is a positive constant depending only on ψ, f and ρ_1. As ε was arbitrarily small, it follows that $\int f\cdot {}^T\overline{L}(\psi)\cdot dxdydu = 0$ for every ψ, namely f is CR on $M\cap\Delta_n(\rho_1)$, as was claimed.

It remains to show that f^d is CR on $M^d\cap\Delta_n(\rho_1)$. First of all, we need some observations. For every compactly supported small deformation M^d stabilizing Φ, the wedge $\mathcal{W}$ attached to $M\backslash\Phi$ is still a wedge attached to $M^d\backslash\Phi$. In addition, this wedge contains a neighborhood of $\big(M^d\backslash\Phi\big)\cap\Delta_n(\rho_1)$ in $\mathbb{C}^n$ by property **(3)** of Proposition 11.4. As Φ was supposed to be stably removable, it follows that there exists a wedge $\mathcal{W}_1$ attached to M^d (including points of Φ) to which holomorphic functions in $\mathcal{W}$ extend holomorphically.

Consequently, replacing $M^d\cap\Delta_n(\rho_1)$ by M, we are led to prove the following lemma, which, on the geometric side, is totally similar to Lemma 11.3, except that the wedge $\mathcal{W}$ attached to $M\backslash\Phi$ appearing in the formulation of Lemma 11.3 is now replaced by a neighborhood Ω of $M\backslash\Phi$ in $\mathbb{C}^n$.

Lemma 11.6. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of CR dimension $m\geq 1$ and of codimension $d = n-m\geq 1$, let $\Phi\subset M$ be a nonempty proper closed subset whose $(2m+d)$-dimensional Hausdorff measure is equal to zero. Let Ω be a neighborhood of $M\backslash\Phi$ in $\mathbb{C}^n$ and let $\mathcal{W}_1$ be a wedge attached to M, including points of Φ. Let $f\in L^1_{loc}(M)$ and assume that its restriction to $M\backslash\Phi$ extends as a holomorphic function $f'\in\mathcal{O}(\Omega\cup\mathcal{W}_1)$. Then f is CR all over M.*

Proof. It suffices to prove that f is CR at every point of Φ. Let $q\in\Phi$ be arbitrary and let $\mathcal{W}_q$ be a local wedge of edge M at q which is contained in $\mathcal{W}_1$. Without loss of generality, we can assume that in coordinates $(z,w) = (x+iy, u+iv)\in\mathbb{C}^m\times\mathbb{C}^d$ vanishing at q with $T_qM = \{v=0\}$, the generic submanifold M is represented locally in the polydisc $\Delta_n(\rho_1)$ by $v = \varphi(x,y,u)$ for some $\mathcal{C}^{2,\alpha}$-smooth $\mathbb{R}^d$-valued mapping φ defined on the real cube on $\mathbb{I}_{2m+d}(\rho_1)$. First of all, we construct a family of analytic discs half attached to M whose interior is contained in the local wedge $\mathcal{W}_q\subset\mathcal{W}_1$.

Lemma 11.7. *There exists a family of analytic discs $A_s(\zeta)$, with $s\in\mathbb{R}^{2m+d-1}$, $|s|\leq 2\delta$ for some $\delta>0$, and $\zeta\in\overline{\Delta}$, which is of class $\mathcal{C}^{2,\alpha-0}$ with respect to all variables, such that*

- **(1)** $A_0(1) = q$.
- **(2)** $A_s(\overline{\Delta})\subset\Delta_n(\rho_1)$.
- **(3)** $A_s(\Delta)\subset\mathcal{W}_q\cap\Delta_n(\rho_1)$.
- **(4)** $A_s(\partial^+\Delta)\subset M$.
- **(5)** $A_s(i)\in M\backslash\Phi$ *and* $A_s(-i)\in M\backslash\Phi$ *for all* s.

(6) *The mapping* $[-2\delta, 2\delta]^{2m+d-1} \times [-\pi/2, \pi/2] \ni (s, \theta) \longmapsto A_s(e^{i\theta}) \in M$ *is an embedding onto a neighborhood of* q *in* M.

(7) *There exists* $\rho_2 > 0$ *such that the image of* $[-\delta, \delta]^{2m+d-1} \times [-\pi/4, \pi/4]$ *through this mapping contains* $M \cap \Delta_n(\rho_2)$.

Proof. Let M^1 be a $\mathcal{C}^{2,\alpha}$-smooth maximally real submanifold of M passing through q such that $M^1 \cap \Phi$ is of zero measure with respect to the Lebesgue measure of M^1. Let $t \in \mathbb{R}^d$ and include M^1 in a parametrized family of maximally real submanifolds M_t^1 which foliates a neighborhood of q in M. Starting with a family of analytic discs $A^1_{c,x,v}(\zeta)$ which are half-attached to M^1 as constructed in Lemma 7.12 above, we first choose the rotation parameter v_0 and a sufficiently small scaling factor c_0 in order that $A^1_{c_0,0,v_0}(\pm i)$ does not belong to Φ. In fact, this can be done for almost every (c_0, v_0), because the mapping $(c, v) \mapsto A^1_{c,0,v}(\pm i)$ is of rank n at every point (c, v) with $c \neq 0$ and $v \neq 0$. In addition, we adjust the rotation parameter v_0 in order that the vector Jv_0 points inside a proper subcone of the cone which defines the wedge $\mathcal{W}_q$. If the scaling parameter c is sufficiently small, this implies that $A^1_{c_0,0,v_0}(\Delta)$ is contained in $\mathcal{W}_q \cap \Delta_n(\rho_1)$, as in Lemma 8.3 above. The translation parameter x runs in $\mathbb{R}^n$ and we may select a $(n-1)$-dimensional parameter subspace x' which is transversal in M^1 to the half boundary $A^1_{c_0,0,v_0}(\partial^+\Delta)$. With such a choice, there exists $\delta > 0$ such that the mapping $[-2\delta, 2\delta]^{n-1} \times [-\pi/2, \pi/2] \ni (x', \theta) \longmapsto A^1_{c_0,x',v_0}(e^{i\theta})$ is a diffeomorphism onto a neighborhood of q in M^1. Finally, using the stability of Bishop's equation under perturbations, we can deform this family of discs by requiring that it is half attached to M_t^1, thus obtaining a family $A_s(\zeta) := A^1_{c_0,x',v_0,t}(\zeta)$ with $s := (x', t) \in \mathbb{R}^{2m+d-1}$. Shrinking δ if necessary, we can check as in the proof of Lemma 8.3 **($\mathbf{9_1}$)** that condition **(5)** holds. This completes the proof. □

Let now $f \in L^1_{loc}(M)$ and let $f' \in \mathcal{O}(\Omega \cup \mathcal{W}_1)$. Thanks to the foliation propery **(6)** of Lemma 11.7, it follows from Fubini's theorem that for almost every translation parameter s, the mapping $e^{i\theta} \mapsto f\left(A_s(e^{i\theta})\right)$ defines a L^1 function on $\partial^+\Delta$. In addition, the restriction of the function $f' \in \mathcal{O}(\Omega \cup \mathcal{W}_1)$ to the disc $A_s(\Delta) \subset \mathcal{W}_q \subset \mathcal{W}_1$ yields a holomorphic function $f'(A_s(\zeta))$ in Δ.

Lemma 11.8. *For almost every* s *with* $|s| \leq 2\delta$, *the function* $f'(A_s(\zeta))$ *belongs to the Hardy space* $H^1(\Delta)$.

Proof. Indeed, for almost every s, the restriction $f\left(A_s(e^{i\theta})\right)$ belongs to $L^1(\partial^+\Delta)$. We can also assume that for almost every s, the intersection $\Phi \cap A_s(\partial^+\Delta)$ is of zero one-dimensional measure. By the assumption of Lemma 11.6, the restriction of $f \circ A_s$ and of $f' \circ A_s$ to $\partial^+\Delta \backslash \Phi$ coincide. Recall that $\partial^-\Delta = \{\zeta \in \partial\Delta : \operatorname{Re}\zeta \leq 0\}$. Since $A_s(\pm i)$ does not belong to Φ and since $A_s\left(e^{i\theta}\right)$ belongs to $\mathcal{W}_q$ for all θ with $\pi/2 < |\theta| \leq \pi$, it follows that $f \circ A_s|_{\partial^+\Delta}$ and $f' \circ A_s|_{\partial^-\Delta}$ (which is holomorphic in a neighborhood of $\partial^-\Delta$ in $\mathbb{C}$) match together in a function which is L^1 on $\partial\Delta$. Let us denote this function by f_s. Furthermore, f_s extends holomorphically to Δ as $f' \circ A_s|_\Delta$. Consequently, $f' \circ A_s|_\Delta$ belongs to the Hardy space $H^1(\Delta)$, which proves the lemma. □

Since we have now established that the boundary value of f' on $M\backslash\Phi$ along the family of discs $A_s(\zeta)$ coincides with f, we can now denote both functions by the same letter f.

For $\varepsilon \geq 0$ small, let now $\chi_\varepsilon\left(s, e^{i\theta}\right)$ be a $\mathcal{C}^2$-smooth function on $[-2\delta, 2\delta] \times \partial\Delta$ which equals ε for $|s| \leq \delta$ and for $\theta \in [-\pi/4, \pi/4]$ and which equals 0 if either $\pi/2 \leq |\theta| \leq \pi$ or $|s| \geq 2\delta/3$. We may require in addition that $||\chi_\varepsilon||_{\mathcal{C}^2} \leq \varepsilon$. We define a deformation

M^ε of M compactly supported in a neighborhood of q by pushing M inside $\mathcal{W}_q$ along the family of discs $A_s(\zeta)$ as follows:

$$M^\varepsilon := \{A_s\left([1-\chi_\varepsilon(s,e^{i\theta})]\,e^{i\theta}\right) : \, |\theta| \leq \pi/2, \ |s| \leq 2\delta\}. \tag{11.9}$$

Notice that M^ε coincides with M outside a small neighborhood of q. Then we have $||M^\varepsilon - M||_{\mathcal{C}^2} \leq C \cdot \varepsilon$, for some constant $C > 0$ which depends only on the $\mathcal{C}^2$ norms of $A_s(\zeta)$ and of $\chi_\varepsilon(s, e^{i\theta})$. If the radius ρ_2 is as in Property **(7)** of Lemma 11.7 above, the deformation $M^\varepsilon \cap \Delta_n(\rho_2)$ is entirely contained in $\mathcal{W}_q$ and since f is holomorphic in $\mathcal{W}_q$, its restriction to $M^\varepsilon \cap \Delta_n(\rho_2)$ is obviously CR. An illustration is provided in the right hand side of the following figure.

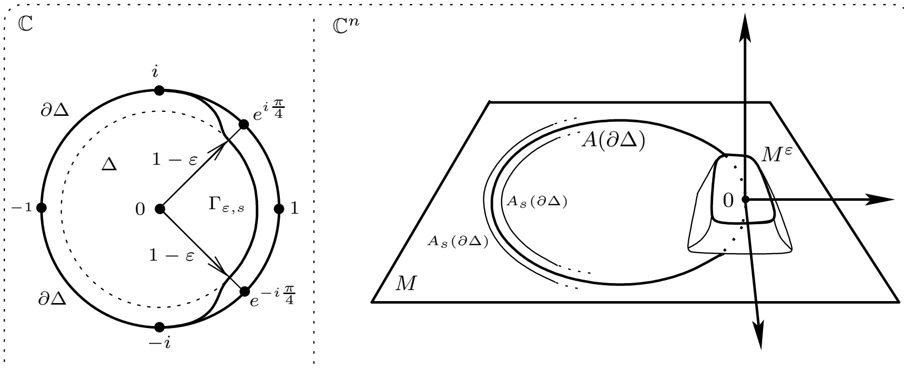

FIGURE 19: THE ARC $\Gamma_{\varepsilon,s}$ AND THE DEFORMATION M^ε

As in [J5], [P1], [MP1], we notice that for every s and every ε, the one-dimensional Lebesgue measure on the arc

$$\Gamma_{\varepsilon,s} := \{[1-\chi_\varepsilon(s,e^{i\theta})]e^{i\theta} \in \Delta : |\theta| \leq \pi\} \tag{11.10}$$

is a Carleson measure. Thanks to the geometric uniformity of these arcs $\Gamma_{\varepsilon,s}$, it follows from an inspection of the proof of L. Carleson's imbedding theorem that there exists a (uniform) constant C such that for all s with $|s| \leq 2\delta$ and all ε, one has the estimate

$$\int_{\Gamma_{\varepsilon,s}} \left| f\left(A_s\left(\left[1-\chi_\varepsilon(s,e^{i\theta})\right]\, e^{i\theta}\right)\right)\right| \cdot d\theta \, \leq \, C \int_{\partial\Delta} |f| \cdot d\theta. \tag{11.11}$$

We are now ready to complete the proof of Lemma 11.6. Let $\pi_{x,y,u}$ denote the projection parallel to the v-space onto the (x,y,u)-space. The mapping $(s,\theta) \mapsto \pi_{x,y,u}\left(A_s(\theta)\right)$ may be used to define new coordinates in a neighborhood of the origin in $\mathbb{C}^m \times \mathbb{R}^d$, an open subset above which M and M^ε are graphed. We shall now work with these coordinates. With respect to the coordinates (s,θ), on M and on M^ε, we have formal adjoints ${}^T\overline{L}$ and ${}^T(\overline{L}^\varepsilon)$ of the basis of CR vector fields with an estimation of the form $\left|\left|{}^T(\overline{L}^\varepsilon) - {}^T\overline{L}\right|\right|_{\mathcal{C}^1} \leq C \cdot \varepsilon$, for some constant $C > 0$. Let now $\psi = \psi(s,\theta)$ be $\mathcal{C}^1$-smooth function with compact support in the set $\{|s| < \delta, \, |\theta| \leq \pi/4\}$. By construction, the subpart of M^ε defined by $\widetilde{M}^\varepsilon := \{A_s\left([1-\chi_\varepsilon(s,e^{i\theta})]\,e^{i\theta}\right) : \, |\theta| \leq \pi/4, \ |s| \leq \delta\}$ is contained in the wedge $\mathcal{W}_q$, hence the restriction of the holomorphic function $f \in \mathcal{W}_q$ to $\widetilde{M}^\varepsilon$ is obviously CR on $\widetilde{M}^\varepsilon$.

For simplicity of notation, we shall denote $f\left(A_s(e^{i\theta})\right)$ by $f_s(\theta)$ and $f\left(A_s\left(\left[1-\chi_\varepsilon(s,e^{i\theta})\right]\, e^{i\theta}\right)\right)$ by $f_s^\varepsilon(\theta)$. Since by construction for every $\varepsilon > 0$,

the L^1 function $(s,\theta) \mapsto f_s^\varepsilon(\theta)$ is annihilated in the distributional sense by the CR vector fields $\overline{L}^\varepsilon$ on $\widetilde{M}^\varepsilon$, we may compute (not writing the arguments (s,θ) of ψ)

$$
\left\{
\begin{aligned}
&\left|\int_{|s|\leq\delta}\int_{|\theta|\leq\pi/4} f_s(\theta)\cdot {}^T\overline{L}(\psi)\cdot dsd\theta\right| = \\
&= \left|\int_{|s|\leq\delta}\int_{|\theta|\leq\pi/4} \Big[f_s(\theta)\cdot {}^T\overline{L}(\psi) - f_s^\varepsilon(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi)\Big]\cdot dsd\theta\right| \\
&\leq \bigg|\int_{|s|\leq\delta}\bigg(\int_{|\theta|\leq\pi/4}\Big[f_s(\theta)\cdot {}^T\overline{L}(\psi) - f_s(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi) + \\
&\qquad\qquad + f_s(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi) - f_s^\varepsilon(\theta)\cdot {}^T(\overline{L}^\varepsilon)(\psi)\Big]\cdot d\theta\bigg)\cdot ds\bigg| \\
&\leq C_1(\psi)\cdot\varepsilon\cdot\int_{|s|\leq\delta}\int_{|\theta|\leq\pi/4}|f_s(\theta)|\cdot dsd\theta + \\
&\qquad + C_2(\psi)\cdot\int_{|s|\leq\delta}\int_{|\theta|\leq\pi/4}|f_s(\theta)-f_s^\varepsilon(\theta)|\cdot dsd\theta \\
&\leq C_1(\psi,f,\delta)\cdot\varepsilon + C_2(\psi,\delta)\cdot\max_{|s|\leq\delta}\int_{|\theta|\leq\pi/4}|f_s(\theta)-f_s^\varepsilon(\theta)|\cdot dsd\theta.
\end{aligned}
\right.
\tag{11.12}
$$

However, thanks to the estimate (11.11) and thanks to Lebesgue's dominated convergence theorem, the last integral tends to zero as ε tends to zero. It follows that the integral in the first line of (11.12) can be made arbitrarily small, hence it vanishes. This proves that f is CR in a neighborhood of q and completes the proof of Lemma 11.6. □

The proof of Lemma 11.3 is complete. □

§12. PROOFS OF THEOREM 1.1 AND OF THEOREM 1.3

12.1. Tree of separatrices linking hyperbolic points. Let $M \subset \mathbb{C}^2$ be a globally minimal $\mathcal{C}^{2,\alpha}$-smooth hypersurface, let $S \subset M$ be a $\mathcal{C}^{2,\alpha}$-smooth open surface (without boundary) and let $K \subset S$ be a proper compact subset of S. Assume that S is totally real outside a discrete subset of complex tangencies which are hyperbolic in the sense of E. Bishop. Since we aim to remove the compact subset K of S, we can shrink the open surface S around K in order that S contains *only finitely many* such hyperbolic complex tangencies, which we shall denote by $\{h_1,\dots,h_\lambda\}$, where λ is some integer, possibly zero. Furthermore, we can assume that ∂S is of class $\mathcal{C}^{2,\alpha}$. As a corollary of the qualitative theory of planar vector fields, due to H. Poincaré and I. Bendixson, we know that

- **(i)** The hyperbolic points $h_1,\dots,h_\lambda$ are singularities of the characteristic foliation $\mathcal{F}_S^c$.
- **(ii)** Incoming to every hyperbolic point $h_1,\dots,h_\lambda$, there are exactly four $\mathcal{C}^{2,\alpha}$-smooth open *separatrices* (to be defined precisely below).
- **(iii)** After perturbing slightly the boundary ∂S if necessary, these separatrices are all transversal to ∂S and the union of all separatrices together with all hyperbolic points makes a *finite tree without cycles in* S (to be defined below).

Precisely, by an (open) *separatrix*, we mean a $\mathcal{C}^{2,\alpha}$-smooth curve $\tau : (0,1) \to S$ with $\frac{d\tau}{ds}(s) \in T_{\tau(s)}S \cap T_{\tau(s)}^cM\backslash\{0\}$ for every $s \in (0,1)$, namely its tangent vectors are all nonzero and characteristic, such that one limit point, say $\lim_{s\to 0}\tau(s)$ is a hyperbolic point, and the other $\lim_{s\to 1}\tau(s)$ either belong to the boundary ∂S or is a second hyperbolic point.

From the local study of saddle phase diagrams (*cf.* [Ha]), we get in addition:

(iv) There exists $\varepsilon > 0$ and for every $l = 1, \dots, \lambda$, there exist two curves $\gamma_l^1, \gamma_l^2 : (-\varepsilon, \varepsilon) \to S$ which are of class $\mathcal{C}^{1,\alpha}$, *not more*, with $\gamma_l^i(0) = h_l$ and $\frac{d\gamma_l^1}{dt}(s) \in T_{\gamma_l(s)} S \cap T^c_{\gamma_l(s)} M \backslash \{0\}$ for every $s \in (-\varepsilon, \varepsilon)$ and for $i = 1, 2$, such that the four open segments $\gamma_l^1(-\varepsilon, 0)$, $\gamma_l^1(0, \varepsilon)$, $\gamma_l^2(-\varepsilon, 0)$ and $\gamma_l^2(0, \varepsilon)$ cover the four pieces of open separatrices incoming at h_l.

Let $\tau_1, \dots, \tau_\mu : (0, 1) \to S$ denote all the separatrices of S, where μ is some integer, possibly equal to zero. By the *finite hyperbolic tree* T_S *of* S, we mean:

$$T_S := \{h_1, \dots, h_\lambda\} \bigcup_{1 \leq k \leq \mu} \tau_k(0, 1). \tag{12.2}$$

We say that T_S has *no cycle* if it does not contain any subset homeomorphic to the unit circle. For instance, in the case where $S \equiv D$ is diffeomorphic to a real disc (as in the assumptions of Theorem 1.1), its hyperbolic tree T_D necessarily has no cycle. However, in the case where S is an annulus (for instance), there is a trivial example of a characteristic foliation with two hyperbolic points and a circle in the hyperbolic tree.

12.3. Hyperbolic decomposition in the disc case. Let the real disc D and the compact subset $K \subset D$ be as in Theorem 1.1. As in §12.1 just above, we shrink D slightly and smooth out its boundary, so that its hyperbolic tree T_D is finite and has no cycle. We may decompose D as the disjoint union

$$D = T_D \cup D_o, \tag{12.4}$$

where the complement of the hyperbolic tree $D_o := D \backslash T_D$ is an open subset of D entirely contained in the totally real part of D. Then D_o has finitely many connected components $D_1, \dots, D_\nu$, the *hyperbolic sectors of* D. Then, for $j = 1, \dots, \nu$, we define the proper closed subsets $C_j := D_j \cap K$ of D_j as illustrated in the left hand side of the following figure.

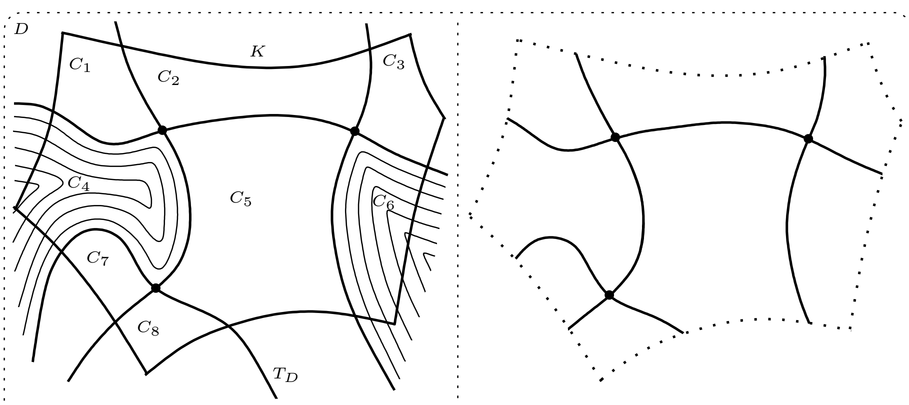

FIGURE 20: REMOVAL OF HYPERBOLIC SECTORS AND REMOVAL OF A TREE WITHOUT CYCLES

Again from H. Poincaré and I. Bendixson's theory, we know that for every component D_j (in which the characteristic foliation is nonsingular), the proper closed subset C_j satisfies the nontransversality condition $\mathcal{F}^c_{D_j}\{C_j\}$ formulated in Theorem 1.2. In FIGURE 20 just above, we have drawn the characteristic curves only for the two sectors D_4 and D_6.

One may observe that $\mathcal{F}_{D_4}\{C_4\}$ and $\mathcal{F}_{D_6}\{C_6\}$ hold true. Also, $K \cap T_D$ is a proper closed subset of the hyperbolic tree of D.

12.5. Global minimality of some complements. Before proceeding to the deduction of Theorems 1.1 and 1.3 from Theorem 1.2, we must verify that the complement $M \backslash K$ is also globally minimal. Here, we state a generalization of Lemma 3.5 to the case where some hyperbolic complex tangencies are allowed. Its proof is not immediate.

Lemma 12.6. *Let M be a $\mathcal{C}^{2,\alpha}$-smooth hypersurface in $\mathbb{C}^2$ and let $S \subset M$ be $\mathcal{C}^{2,\alpha}$-smooth surface which is totally real outside a discrete subset of hyperbolic complex tangencies. Assume that the hyperbolic tree T_S of S has no cycle. Then for every compact subset $K \subset S$ and for an arbitrary point $p \in M \backslash K$, its CR orbit in $M \backslash K$ coincides with its CR orbit in M, minus K, namely*

$$\mathcal{O}_{CR}(M \backslash K, p) = \mathcal{O}_{CR}(M, p) \backslash K. \tag{12.7}$$

Proof. Of course, we may assume that S coincides with the shrinking of a slightly larger surface and has finitely many hyperbolic points $\{h_1, \ldots, h_\lambda\}$, as described in §12.1, with the same notation. Let $K_{T_S} := K \cap T_S$ be the track of K on the hyperbolic tree T_S. Since the intersection of K_{T_S} with any open separatrix may in general coincide with any arbitrary closed subset of an interval, in order to fix ideas, it will be convenient to deal with an enlargement $\overline{K}$ of K_{T_S}, simply defined by filling the possible holes of K_{T_S} in T_S: more precisely, $\overline{K}$ should contain all hyperbolic points together with all separatrices joining them and for every separatrix $\tau_k(0, 1)$ with right limit point $\lim_{s \to 1} \tau_k(s)$ belonging to the boundary of S, we require that $\overline{K}$ contains the segment $\tau_k[0, r_1]$, where $r_1 < 1$ is close enough to 1 in order that $\overline{K}$ effectively contains K_{T_S}.

Obviously, from the inclusions

$$K_{T_S} \subset \overline{K} \subset K, \tag{12.8}$$

we deduce that for every point $p \in M \backslash K$, we have the reverse inclusions

$$\mathcal{O}_{CR}(M \backslash K, p) \subset \mathcal{O}_{CR}(M \backslash \overline{K}, p) \subset \mathcal{O}_{CR}(M \backslash K_{T_S}, p). \tag{12.9}$$

The main step in the proof of Lemma 12.6 will be to establish the following two assertions, implying the third, desired assertion, already stated as (12.7).

(A1) For every point $q \in M \backslash \overline{K}$, we have $\mathcal{O}_{CR}(M \backslash \overline{K}, q) = \mathcal{O}_{CR}(M, q) \backslash \overline{K}$.
(A2) For every point $r \in M \backslash K_{T_S}$, we have $\mathcal{O}_{CR}(M \backslash K_{T_S}, r) = \mathcal{O}_{CR}(M, r) \backslash K_{T_S}$.

Indeed, taking these two assertions for granted, let us conclude the proof of Lemma 12.6. Let $p \in M \backslash K$ and decompose K as a disjoint union $K = K_{T_S} \cup C'$, where $C' := K \backslash K_{T_S}$ is a relatively closed subset of the hypersurface $M' := M \backslash K_{T_S}$. Notice that C' is contained in the totally real part of S. Again thanks to foliation theory, we see that the assumption that K_{T_S} does not contain any cycle entails that C' does not contain maximal characteristic lines of the totally real surface $S \backslash K_{T_S}$. Consequently, all the assumptions of Lemma 3.5 are satisfied, hence by applying it to p, we deduce that $\mathcal{O}_{CR}(M' \backslash C', p) = \mathcal{O}_{CR}(M', p) \backslash C'$. By developing in length this identity between sets, we get

$$\begin{aligned} \mathcal{O}_{CR}(M \backslash K, p) &= \mathcal{O}_{CR}\left((M \backslash K_{T_S}) \backslash C', p\right) \\ &= \left[\mathcal{O}_{CR}(M \backslash K_{T_S}, p)\right] \backslash C' \\ &= \left[\mathcal{O}_{CR}(M, p) \backslash K_{T_S}\right] \backslash C' \\ &= \mathcal{O}_{CR}(M, p) \backslash K, \end{aligned} \tag{12.10}$$

where, for the passage from the second to the third line, we use **(A2)**. This is (12.7), as desired.

Thus, the (main) remaining task is to establish the assertion **(A2)**, with **(A1)** being a preliminary step.

First of all, we show how to deduce **(A2)** from **(A1)**. Pick an arbitrary point $r \in M \backslash K_{T_S}$. Of course, we have the trivial inclusion $\mathcal{O}_{CR}(M \backslash K_{T_S}, r) \subset \mathcal{O}_{CR}(M, r) \backslash K_{T_S}$ and we want an equality. As $\overline{K}$ contains K_{T_S}, we have either $r \in M \backslash \overline{K}$ (first case) or $r \in \overline{K} \backslash K_{T_S}$ (second case), *see* FIGURE 21 just below, where r is located in $\overline{K} \backslash K_{T_S}$.

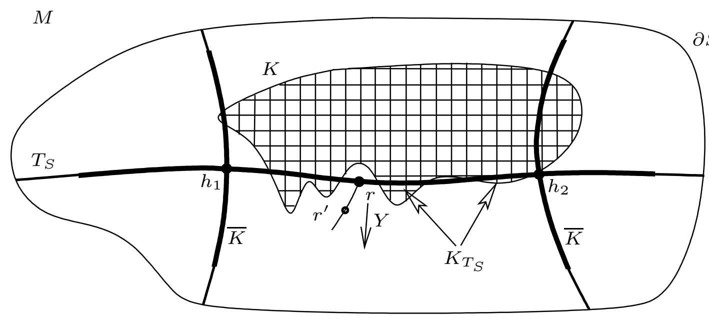

FIGURE 21: THE DISPOSITION OF K, K_{T_S} AND $\overline{K}$

Since the second case makes it impossible to apply **(A1)**, we need to find another point r' in the CR orbit of r in $M \backslash K_{T_S}$ such that r' belongs to $M \backslash \overline{K}$. This is elementary. We make a dichotomy: either $r = h_l$ is a hyperbolic point or it belongs to an open separatrix $\tau_k(0, 1)$. If $r = h_l \in M \backslash K_{T_S}$ is a hyperbolic point, we may use one of the two $\mathcal{C}^{2,\alpha}$-smooth complex tangent curves γ_l^1 or γ_l^2 passing through h_l and running in $M \backslash K_{T_S}$ to join r with another point which belongs to an open separatrix and which obviously lies in the same CR orbit $\mathcal{O}_{CR}(M \backslash K_{T_S}, r)$. Hence, we may assume that $r \in \overline{K} \backslash K_{T_S}$ belongs to an open separatrix. Since S is now maximally real near r, we may choose a $T^c M$-tangent vector field Y defined in a neighborhood of r which is transversal to S at r. Then for all $\delta > 0$ small enough, the point $r' := \exp(\delta Y)(r)$ is outside S, hence does not belong to $\overline{K}$ and clearly lies in the same CR orbit $\mathcal{O}_{CR}(M \backslash K_{T_S}, r)$.

In summary, when $r \in \overline{K} \backslash K_{T_S}$, we have exhibited a point $r' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r)$ with $r' \in M \backslash \overline{K}$ so that it suffices now to show that for every point $r \in M \backslash \overline{K}$, we have $\mathcal{O}_{CR}(M \backslash K_{T_S}, r) = \mathcal{O}_{CR}(M, r) \backslash K_{T_S}$.

Using a trivial inclusion and applying **(A1)**, we deduce that

$$\mathcal{O}_{CR}(M \backslash K_{T_S}, r) \supset \mathcal{O}_{CR}(M \backslash \overline{K}, r) = \mathcal{O}_{CR}(M, r) \backslash \overline{K}. \tag{12.11}$$

Unfortunately, there may well exist points of $\mathcal{O}_{CR}(M, r)$ belonging to $\overline{K} \backslash K_{T_S}$, so that it remains to show that *every* point $r' \in \mathcal{O}_{CR}(M, r) \cap [\overline{K} \backslash K_{T_S}]$ also belong to $\mathcal{O}_{CR}(M \backslash K_{T_S}, r)$. Again, this last step is elementary and totally analogous to the above argument: we first claim that we can join such a point r' to a point $r''' \in M \backslash \overline{K}$ by means of a piecewise smooth CR curve running in $M \backslash K_{T_S}$. Indeed, if $r' = h_l$ is a hyperbolic point, we may first use one of the two $\mathcal{C}^{2,\alpha}$-smooth complex tangent curves γ_l^1 or γ_l^2 passing through h_l and running in $M \backslash K_{T_S}$ to join r' with another nearby point r'' which belongs to an open separatrix. If r' already belongs to an open separatrix, we simply set $r'' := r'$. Since S is now maximally real near r'', we may choose a $T^c M$-tangent vector field Y defined in a neighborhood of r'' which is transversal to S at r''. Then for all

$\delta > 0$ small enough, the point $r''' := \exp(\delta Y)(r'')$ satisfies $r''' \notin S$, whence $r''' \notin \overline{K}$. Of course, by choosing r'' and r''' sufficiently close to r', it follows that the piecewise smooth CR curve joining them does not meet K_{T_S}. We deduce that $r' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r''')$.

Since by assumption $r' \in \mathcal{O}_{CR}(M, r)$, it follows that $r''' \in \mathcal{O}_{CR}(M, r)$ and then $r''' \in \mathcal{O}_{CR}(M, r) \backslash \overline{K}$. By means of the supclusion (12.11), we deduce that $r''' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r)$.

Finally, from the two relations $r''' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r)$ and $r' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r''')$, we conclude immediately that $r' \in \mathcal{O}_{CR}(M \backslash K_{T_S}, r)$. We have thus shown that the supclusion in (12.11) is an equality.

This completes the deduction of **(A2)** from **(A1)**.

It remains now to establish **(A1)**. We remind that in Section 3, we derived Lemma 3.5 from Lemma 3.7. By means of a totally similar argument, which we shall not repeat, one deduces **(A1)** from the following assertion. Remind that as M is a hypersurface in $\mathbb{C}^2$, its CR orbits are of dimension either 2 or 3.

Lemma 12.12. *Let M, S, T_S and $\overline{K} \subset T_S$ be as above. There exists a connected submanifold Ω embedded in M containing the hyperbolic tree T_S such that*

- **(1)** *Ω is a T^cM-integral manifold, namely $T_p^cM \subset T_p\Omega$ for all $p \in \Omega$.*
- **(2)** *Ω is contained in a single CR orbit of M.*
- **(3)** *$\Omega \backslash \overline{K}$ is also contained in a single CR orbit of $M \backslash \overline{K}$.*

More precisely, Ω is an open neighborhood of T_S if it is of real dimension 3 and a complex curve surrounding T_S if it is of dimension 2.

Proof. We shall construct Ω by means of a flowing procedure, starting from a local piece of it. We start locally in a neighborhood of a fixed point $p_0 \in \overline{K} \backslash \{h_1, \dots, h_\lambda\}$, whose precise choice does not matter. Since S is totally real in a neighborhood of p_0, there exists a locally defined T^cM-tangent vector field Y which is transversal to S at p_0. Consequently, for $\delta > 0$ small enough, the small segment $I_0 := \{\exp(sY)(p_0) : -\delta < s < \delta\}$ is transversal to S at p_0 and moreover, the two half-segments

$$I_0^{\pm} := \{\exp(sY)(p_0) : 0 < \pm s < \delta\} \tag{12.13}$$

lie in $M \backslash S$. Since p_0 belongs to some T^cM-tangent open separatrix $\tau_k(0,1)$, there exists a $\mathcal{C}^{1,\alpha}$-smooth vector field X defined in a neighborhood of p_0 in M which is tangent to S and whose integral curve passing through p_0 is a piece of $\tau_k(0,1)$. Since Y is transversal to S at p_0, it follows that the set $\omega_0 := \{\exp(s_2X)(\exp(s_1Y)(p_0)) : -\delta < s_1,\, s_2 < \delta\}$ is a well-defined $\mathcal{C}^{1,\alpha}$-smooth codimension one small submanifold passing through p_0 which is transversal to S at p_0. Clearly, we even have $T_{p_0}\omega_0 = T_{p_0}^cM$. Thanks to the fact that the flow of X stabilizes S, we see that the integral curves $s_2 \mapsto \exp(s_2X)(\exp(s_1Y)(p_0))$ are contained in $M \backslash S$ for every starting point $\exp(s_1Y)(p_0)$ in the segment I_0 which does not lie in S, namely for all $s_1 \neq 0$. We deduce that the two open halves of ω_0 defined by

$$\omega_0^{\pm} := \{\exp(s_2X)(\exp(s_1Y)(p_0)) : 0 < \pm s_1 < \delta,\, -\delta < s_2 < \delta\} \tag{12.14}$$

are contained in a single CR orbit of $M \backslash \overline{K}$.

To begin with, assume that the CR orbit in $M \backslash \overline{K}$ the point $q_0^+ := \exp\left(\frac{\delta}{2}Y\right)(p_0)$ which belong to ω_0^+, as drawn in FIGURE 22 below, is of real dimension 2. Afterwards, we shall treat the case where its CR orbit in $M \backslash \overline{K}$ is of dimension 3.

Since M is a hypersurface in $\mathbb{C}^2$ and since we have just proved that the CR orbit $\mathcal{O}_{CR}(M \backslash \overline{K}, q_0^+)$ already contains the 2-dimensional half piece ω_0^+, we deduce that ω_0^+

is a piece of complex curve whose boundary $\partial\omega_0^+$ is (by construction) contained in the separatrix $\tau_k(0,1)$. Since $\tau_k(0,1)$ is an embedded segment, we may suppose from the beginning that the vector field X is defined in a neighborhood of $\tau_k(0,1)$ in M. Using then the flow of X, we may easily prolong the small piece ω_0^+ to get a semi-local $\mathcal{C}^{1,\alpha}$-smooth submanifold ω_k^+ stretched along $\tau_k(0,1)$, which constitutes its boundary. Again, this piece ω_k^+ is (by construction) contained in the CR orbit of q_0^+ in $M\backslash\overline{K}$. By the fundamental stability property of CR orbits under flows, we deduce that ω_k^+ is in fact a piece of complex curve with boundary $\tau_k(0,1)$.

Remind that by definition of separatrices, the point $\tau_k(0)$ is always a hyperbolic point. There is a dichotomy: either $\tau_k(1)$ is also a hyperbolic point or it lies in ∂S. If $\tau_k(1)$ is a hyperbolic point, then by the definition of $\overline{K}$, the complete boundary $\partial\omega_k^+ = \tau_k(0,1)$ is contained in $\overline{K}$, hence it may *not* be crossed by means of a CR curve running in $M\backslash\overline{K}$. Since the piece ω_k^+ will be flowed all around T_S, our filling $\overline{K}$ of K_{T_S} was motivated by the desire of simplifying the geometric situation without having to discuss whether K_{T_S} contains or does not contain the whole segment $\tau_k(0,1)$, for each $k = 1, \dots, \mu$.

Before studying the case where $\tau_k(1) \in \partial S$, let us analyze the local situation in a neighborhood of the hyperbolic point $\tau_k(0) =: h_l$, for some l with $1 \leq l \leq \lambda$.

As a preliminary, in order to understand clearly the situation, let us assume that the two characteristic curves γ_l^1 and γ_l^2 passing through h_l are of class $\mathcal{C}^{2,\alpha}$, an assumption which would be satisfied if we had assumed that M and S are of class $\mathcal{C}^{3,\alpha}$. By a straightening of γ_l^1 and of γ_l^2, which induces a loss of one derivative, we easily show that there exist two linearly independent $\mathcal{C}^{1,\alpha}$-smooth T^cM-tangent vector fields X_1 and X_2 whose integral curves issued from h_l coincide with γ_l^1 and γ_l^2. After possibly renumbering and reversing X_1 and X_2 and also reparametrizing γ_l^1 and γ_l^2, we may assume that $\gamma_l^1(s) = \exp(sX_1)(h_l)$ and that $\gamma_l^2(s) = \exp(sX_2)(h_l)$, for all small $s > 0$. Furthermore, we may assume that the direction of X_2 in a neighborhood of h_l is the same as the direction from $\partial\omega_k^+$ to ω_k^+.

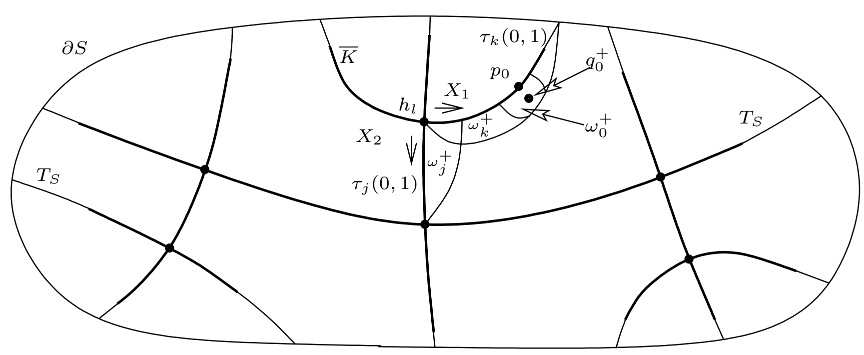

FIGURE 22: ORBIT NEIGHORHOOD OF THE COMPACT SET K_{T_S}

As in FIGURE 22 just above, let $\tau_j(0,1)$ be the separatrix issued from h_l in the positive direction of X_2. We may assume that $\tau_j(0) = h_l$. Thanks to the flow of the vector field X_2, we may now propagate the piece of complex curve ω_k^+ by stretching it along $\tau_j(0,1)$ in a neighborhood of h_l. Using then the flow of a semi-locally defined complex tangent vector field defined in a neighborhood of $\tau_j(0,1)$, we may extend this local piece as a

complex curve ω_j^+ with boundary $\tau_j(0,1)$. Finally, ω_k^+ and ω_j^+ glue together as a complex curve with boundary $\tau_k(0,1) \cup \{h_l\} \cup \tau_j(0,1)$ and corner h_l.

However, by **(iv)** above, γ_l^1 and γ_l^2 are only of class $\mathcal{C}^{1,\alpha}$. Examples for which this regularity is optimal are easily found. Straightening them is again possible, but the vector fields X_1 and X_2 would be of class $\mathcal{C}^\alpha$, and we would lose the uniqueness of their integral curves as well as the regularity of their flow. Consequently, to prove that ω_k^+ propagates along the second separatrix, with its boundary contained in it, we must proceed differently : the proof is longer and we need one more diagram.

In FIGURE 23 just below, we draw the sadlle-looking surface S in the 3-dimensional space M ; the horizontal plane passing through h_l is thought to be the complex tangent plane $T^c_{h_l} M$.

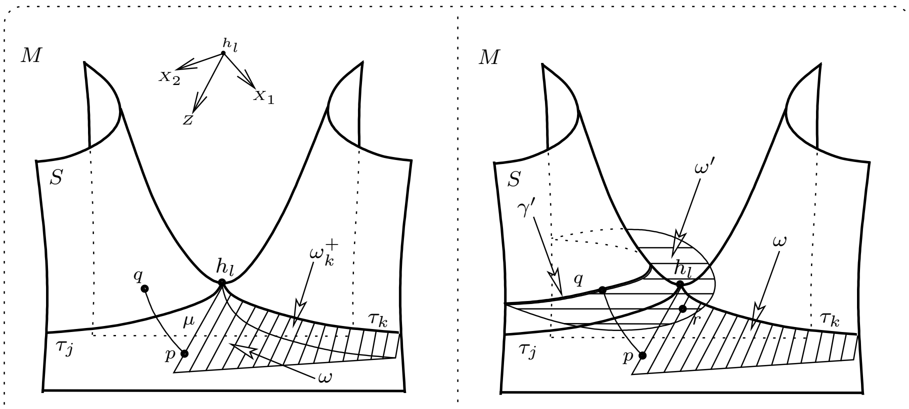

FIGURE 23: BEHAVIOUR OF THE COMPLEX CURVE ORBIT AROUND A HYPERBOLIC POINT

Let us introduce two T^cM-tangent vector fields X_1 and X_2 defined in a neighborhood of h_l with $X_1(h_l)$ directed along τ_k in the sense of increasing s and $X_2(h_l)$ directed along τ_j in the sense of increasing s. Let Z denote the vector field $X_1 + X_2$, as shown in the top of the left hand side of FIGURE 23 above. Using the flow of Z we can begin by extending the banana-looking piece ω_k^+ of complex curve by introducing the submanifold ω consisting of points

$$\exp(s_2 Z)(\tau_k(s_1)), \tag{12.15}$$

where $0 < s_1 < \delta$ and $0 < s_2 < \delta$, for some small $\delta > 0$. One checks that all these points stay in $M \backslash S$, hence are contained in the same CR orbit as ω_k^+ in $M \backslash \overline{K}$. By the stability property of CR orbits, it follows of course that ω is a piece of complex curve contained in M.

For $0 < s < \delta$, let $\mu(s) := \exp(sZ)(h_l)$ denote the CR curve lying "between" τ_k and τ_j and which constitutes a part of the boundary of ω. Let p be an arbitrary point of this curve, close to h_l.

Lemma 12.16. *The integral curve $s \mapsto \exp(-sX_1)(p)$ of $-X_1$ issued from p necessarily intersects S at a point q close to h_l and close to τ_j (cf.* FIGURE 23).

Proof. First of all, we need some preliminary.

Thanks to the existence of a "$1/8$ piece" of complex curve ω with $h_l \in \overline{\omega}$ which is contained in the hypersurface M, we see that M is necessarily Levi-degenerate at h_l.

Next, we introduce local holomorphic coordinates $(z, w) = (x + iy, u + iv) \in \mathbb{C}^2$ vanishing at h_l in which the hypersurface M is given as the graph $v = \varphi(x, y, u)$, where φ is a $\mathcal{C}^{2,\alpha}$-smooth function. Since M is Levi degenerate at h_l, we may assume that $\varphi(x, y, u)| \leq C \cdot (|x| + |y| + |u|)^{2+\alpha}$. We may also assume that the surface S, as a subset of M, is represented by one supplementary equation of the form $u = h(x, y)$, where the $\mathcal{C}^{2,\alpha}$-smooth function h satisfies

$$\begin{cases} h(x,y) & = z\bar{z} + \gamma\,(z^2 + \bar{z}^2) + \mathrm{O}\left(|z|^{2+\alpha}\right) \\ & = (2\gamma + 1)\, x^2 - (2\gamma - 1)\, y^2 + \mathrm{O}\left(|z|^{2+\alpha}\right), \end{cases} \tag{12.17}$$

and where $\gamma > \frac{1}{2}$ is E. Bishop's invariant. Then the tangents at h_l to the two half-separatrices τ_k and τ_j are given respectively by the linear (in)equations $x > 0$, $y = -\frac{2\gamma+1}{2\gamma-1}\, x$, $u = 0$ and $x < 0$, $y = -\frac{2\gamma+1}{2\gamma-1}\, x$, $u = 0$. In FIGURE 23, where we do not draw the axes, the u-axis is vertical, the y axis points behind h_l and the x-axis is horizontal, from left to right.

Expressing the two T^cM-tangent vector fields X_1 and X_2 in the (natural) real coordinates (x, y, u) over M, we may write them as

$$\begin{cases} X_1 & = \dfrac{\partial}{\partial x} - \left(\dfrac{2\gamma+1}{2\gamma-1}\right) \dfrac{\partial}{\partial y} + A_1(x,y,u)\, \dfrac{\partial}{\partial u}, \\ X_2 & = -\dfrac{\partial}{\partial x} - \left(\dfrac{2\gamma+1}{2\gamma-1}\right) \dfrac{\partial}{\partial y} + A_2(x,y,u)\, \dfrac{\partial}{\partial u}. \end{cases} \tag{12.18}$$

Since φ vanishes to second order at h_l, the two $\mathcal{C}^{1,\alpha}$-smooth coefficients A_1 and A_2 satisfy an estimate of the form

$$|A_1, A_2(x, y, u)| < C \cdot (|x| + |y| + |u|)^{1+\alpha}\,. \tag{12.19}$$

Now, we come back to the integral curve of Lemma 12.16. It is contained in the real 2-surface passing through h_l defined by

$$\Sigma := \{\exp(-s_2 X_1)(\exp(s_1 Z)(h_l)) : \, -\delta < s_1, s_2 < \delta\}, \tag{12.20}$$

for some $\delta > 0$. Because the vector fields X_1, X_2 and $Z = X_1 + X_2$ have $\mathcal{C}^{1,\alpha}$-smooth coefficients, the surface Σ is only $\mathcal{C}^{1,\alpha}$-smooth in general. In M equipped with the three real coordinates (x, y, u), we may parametrize Σ by a mapping of the form

$$(s_1, s_2) \longmapsto \left(s_2 - 2s_1 \left(\frac{2\gamma+1}{2\gamma-1}\right),\ s_2 \left(\frac{2\gamma+1}{2\gamma-1}\right),\ u(s_1, s_2)\right), \tag{12.21}$$

where u is of class $\mathcal{C}^{1,\alpha}$. It is clear that $u(0) = \partial_{s_1} u(0) = \partial_{s_2} u(0) = 0$, so that there is a constant C such that

$$|u(s_1,\ s_2)| < C \cdot (|s_1| + |s_2|)^{1+\alpha}\,, \tag{12.22}$$

since u is of class $\mathcal{C}^{1,\alpha}$. Furthermore, by inspecting the flows appearing in (12.20), taking account of the estimates (12.19), we claim that u satisfies the better estimate

$$|u(s_1, s_2)| < C \cdot (|s_1| + |s_2|)^{2+\alpha}\,, \tag{12.23}$$

for some constant $C > 0$. In other words, Σ osculates the complex tangent plane $T^c_{h_l}M$ to second order at h_l : Σ is more flat than S at h_l. One may check the estimate (12.23) is sufficient to establish Lemma 12.16, because the second jet of the saddle function $h(x, y)$ does not vanish at the origin.

To prove the claim, we formulate the main argument as an independent assertion. Mild modifications of this argument apply to our case, but we shall not provide all the details.

Let $L_1 := \frac{\partial}{\partial x} + A_1(x,y,u)\,\frac{\partial}{\partial u}$ and $L_2 := \frac{\partial}{\partial y} + A_2(x,y,u)\,\frac{\partial}{\partial u}$ be two vector fields having $\mathcal{C}^{1,\alpha}$-smooth coefficients satisfying (12.19). Denote by $s_1 \longmapsto (s_1, \lambda(s_1), \mu(s_1))$ the integral curve of L_1 passing through the origin. It is $\mathcal{C}^{2,\alpha}$-smooth and we have

$$|\lambda(s_1)| < C\cdot|s_1|^{2+\alpha} \qquad \text{and} \qquad |\mu(s_1)| < C\cdot|s_1|^{2+\alpha}, \tag{12.24}$$

for some constant $C > 0$. Consider the composition of flows $\exp(s_2 L_2)(\exp(s_1 L_1)(0))$. We have to solve the system of ordinary differential equations

$$\frac{dx}{ds_2} = 0, \qquad \frac{dy}{ds_2} = 1, \qquad \frac{du}{ds_2} = A_2(x,y,u) \tag{12.25}$$

with initial conditions

$$x(0) = s_1, \qquad y(0) = \lambda(s_1), \qquad u(0) = \mu(s_1). \tag{12.26}$$

This yields $x(s_1,s_2) = s_1$, $y(s_1,s_2) = \lambda(s_1) + s_2$ and the integral equation

$$u(s_1,s_2) = \mu(s_1) + \int_0^{s_2} A_2\left(s_1, s_2' + \lambda(s_1), u(s_1,s_2')\right)\, ds_2'. \tag{12.27}$$

Since u is at least $\mathcal{C}^{1,\alpha}$-smooth and vanishes to order 1 at $(s_1,s_2) = (0,0)$, we know already that it satisfies (12.22). Using (12.19), it is now elementary to provide an upper estimate of the right hand side of (12.27) which yields the desired estimate (12.23).

The proof of Lemma 12.16 is complete. □

So, for various points $p = \mu(s)$ close to h_l the intersection points $q \in S$ exist. If all points q belong to τ_j, we are done: the piece ω extends a $1/4$ piece of complex curve with boundary $\tau_k \cup \tau_j$ near h_l and corner h_l.

Assume therefore that one such point q does not belong to τ_j, as drawn in the left hand side of FIGURE 23 above. Suppose that q lies above τ_j, the case where q lies under τ_j being similar and in fact simpler. The characteristic curve $\gamma' \subset S$ passing through q stays above τ_j and is nonsingular. Propagating the complex curve ω in $M\backslash\overline{K}$ by means of the flow of $-X_1$, we deduce that there exists at q a local piece ω_q^+ of complex curve with boundary contained in γ' which is contained in the same CR orbit as ω. Using then the flow of a CR vector having γ' as an integral curve, we can propagate ω_q^+ along γ', which yields a long thin banana-looking complex curve with boundary in γ'. However, this piece may remain too thin. Fortunately, thanks to the flow of $X_1 - X_2$, we can extend it as a piece ω' of complex curve with boundary γ' which goes over h_l, with respect to a complex projection onto $T_{h_l}^c M$, as illustrated in FIGURE 23 above. We claim that this yields a contradiction.

Indeed, as ω and ω' are complex curves, they are locally defined as graphs of holomorphic functions g and g' defined in domains D and D' in the complex line $T_{h_l}^c M$. By construction, there exists a point in $r \in D \cap D'$ at which the values of g and g' are distinct. However, since by construction g and g' coincide in a neighborhood of the CR curve joining p to q, they must coincide at r because of the principle of analytic continuation: this is a contradiction. In conclusion, the CR orbit passes through the hyperbolic point h_l, in a neighborhood of which it consists of a cornered complex curve with boundary $\tau_k \cup \tau_j$.

We can now continue the proof. Since the hyperbolic tree T_S does not contain any cycle, by proceeding this way we claim that the small piece of complex curve ω_0^+ propagates all around T_S and matches up as a smooth complex curve Ω containing the hyperbolic tree. Indeed, in the case where $\tau_k(1)$ is not a hyperbolic point, recall that we arranged at the beginning that $\overline{K} \cap \tau_k(0,1) = \tau_k(0, r_1]$, where $r_1 < 1$. It is then crucial that when a limit point $\tau_k(1)$ belongs to ∂S, we escape from $\overline{K}$ and using a local CR vector field Y

transversal to S, we may cross the separatrix $\tau_k(0,1)$ at a point $\tau_k(r_2)$ where r_2 satisfies $r_1 < r_2 < 1$. Hence, we pass to the other side of S in M and then, by means of a futher flowing, we turn around to the other side of $\tau_k(0,1)$. Also, the two pieces in either side of $\tau_k(0,1)$ match up at least $\mathcal{C}^{1,\alpha}$-smoothly. Then thanks to the stability property of orbits under flows, we deduce that these two pieces match up as a piece of complex curve containing $\tau_k(0,1)$ in its interior.

We thus construct the complex curve Ω surrounding T_S, which is obviously contained in a single CR orbit of M. Also, by construction, $\Omega\backslash\overline{K}$ is contained in a single CR orbit of $M\backslash\overline{K}$. Thus, we have established Lemma 3.12 under the assumption that the CR orbit of q_0^+ is two-dimensional.

Assume finally that the CR orbit of q_0^+ is 3-dimensional. By a similar propagation procedure, we easily construct a neighborhood Ω in M of the hyperbolic tree satisfying conditions **(1)**, **(2)** and **(3)** of Lemma 12.12. This complete its proof. □

The proof of Lemma 12.6 is complete. □

12.28. Proofs of Theorems 1.1 and 1.3. We can now prove Theorem 1.1. In fact, we shall directly prove the more general version stated as Theorem 1.3 which implies Theorem 1.1 as a corollary, thanks to the geometric observations of §12.3.

First of all, we notice that as $M\backslash K$ is globally minimal, there exists a wedge attached to $M\backslash K$ to which continuous CR functions on $M\backslash K$ extend holomorphically. Hence, the CR-removability of K is a consequence of its $\mathcal{W}$-removability. Also, Lemma 11.3 shows that the L^{p}-removability of K is a consequence of its $\mathcal{W}$-removability. Consequently, it suffices to establish that K is $\mathcal{W}$-removable in Theorem 1.3.

Let T_S be the hyperbolic tree of (a suitable shrinking of) S, which contains no cycle by assumption. Let ω_1 be a one-sided neighborhood of $M\backslash K$ in $\mathbb{C}^2$. Because the nontransversality condition $\mathcal{F}^c_{S\backslash T_S}\{K\cap(S\backslash T_S)\}$ holds true by assumption, we may apply Theorem 1.2 to the totally real surface $S\backslash T_S$ in the globally minimal (thanks to Lemma 12.6) hypersurface $M\backslash K_{T_S}$ to remove the proper closed subset $K\cap(S\backslash T_S)$. We deduce that there exists a one-sided neighborhood ω_2 of $M\backslash K_{T_S}$ in $\mathbb{C}^2$ such that (after shrinking ω_1 if necessary), holomorphic functions in ω_1 extend holomorphically to ω_2. Then we slightly deform M inside ω_2 over points of $K\cap(S\backslash T_S)$. We obtain a $\mathcal{C}^{2,\alpha}$-smooth hypersurface M^d with $M^d\backslash K_{T_S}\subset\omega_2$. Also, by stability of global minimality under small perturbations, we can assume that M^d is also globally minimal. By construction, we obtain holomorphic functions in the neighborhood ω_2 of $M^d\backslash K_{T_S}$ in $\mathbb{C}^2$.

Since M and M^d are of codimension 1, the union of a one-sided neighborhood ω^d of M^d in $\mathbb{C}^2$ together with ω_2 constitutes a one-sided neighborhood of M in $\mathbb{C}^2$. To conclude the proof of Theorem 1.1, it suffices therefore to show that the closed set K_{T_S} is $\mathcal{W}$-removable. The reader is referred to FIGURE 20 above for an illustration.

Reasoning by contradiction (as for the proof of Theorem 1.2'), let $K_{\rm nr}\subset K_{T_S}$ denote the smallest nonremovable subset of K_{T_S}. If $K_{\rm nr}$ is empty, we are done, gratuitously. Assume therefore that $K_{\rm nr}$ is nonempty. Let T' be a connected component of the minimal subtree of T containing $K_{\rm nr}$. By a *subtree* of a tree T defined as in (12.2) above, we mean of course a finite union of some of the separatrices $\tau_k(0,1)$ together with all hyperbolic points which are endpoints of separatrices. Since T' does not contain any subset homeomorphic to the unit circle, there exists at least one extremal branch of T', say $\tau_1(0,1)$ after renumbering, with $\tau_1(1)\in\partial S$. To reach a contradiction, we shall show that at least one point of the nonempty set $K_{\rm nr}\cap T'$ is in fact $\mathcal{W}$-removable.

If the subtree T' consists of the single branch $\tau_1(0,1)$ together with the single elliptic point $\tau_1(0)$, thanks to properties **(iii)** and **(iv)** of §12.1, we can enlarge a little bit this branch by prolongating the curve $\tau_1(0,1)$ to an open $\mathcal{C}^{2,\alpha}$-smooth Jordan arc $\tau_1(-\varepsilon, 1+\varepsilon)$, for some $\varepsilon > 0$, with the appendix $\tau_1(1, 1+\varepsilon)$ outside S (in the slightly larger surface containing S). Then we remind that by a special case of Theorem 4 (ii) of [MP1], every proper closed subset of $\tau_1(-\varepsilon, 1+\varepsilon)$ is $\mathcal{W}$-removable. It follows that the proper closed subset $K_{\rm nr}$ of the Jordan arc $\tau_1(-\varepsilon, 1+\varepsilon)$ is removable, which yields the desired contradiction in the case where T' consists of a single branch together with a single elliptic point.

If T' consists of at least two branches, again with $\tau_1(1) \in \partial S$, then applying Theorem 4 (ii) of [MP1], we may at least deduce that $K_{\rm nr} \cap \tau_1(0,1)$ is $\mathcal{W}$-removable, since $K_{\rm nr} \cap \tau_1(0,1)$ is contained in $\tau_1(0, r_1]$, for some $r_1 < 1$. But possibly, this set $K_{\rm nr} \cap \tau_1(0,1)$ could be empty.

However, we claim that it is nonempty. Indeed, otherwise, if $K_{\rm nr} \cap \tau_1[0,1)$ consists of the single point $\tau_1(0)$, which is a hyperbolic point, then $K_{\rm nr}$ is in fact contained in the smaller subtree T'' defined by $T'' := T' \backslash \tau_1(0,1)$ (here, we use that $\tau_1(0)$ is a hyperbolic point, hence there exists another branch $\tau_k(0,1)$ with $\tau_k(1) = \tau_1(0)$ or $\tau_k(0) = \tau_1(0)$). This contradicts the assumption that T' is the minimal subtree containing $K_{\rm nr}$. Then $K_{\rm nr} \cap \tau_1(0,1)$ is nonempty and removable, which contradict the assumption that $K_{\rm nr}$ is the smallest nonremovable subset of K.

The proofs of Theorems 1.1 and 1.3 are complete.

§13. Applications to the edge of the wedge theorem

In this section we formulate three versions of the edge of the wedge theorem for holomorphic and meromorphic functions, two of which are based upon an application of our removable singularities theorems. Let us begin with some definition.

13.1. Preliminary. Let E be a generic submanifold of $\mathbb{C}^n$, which may be maximally real. By a *double wedge attached to* E, we mean a pair $(\mathcal{W}_1, \mathcal{W}_2)$ of disjoint wedges attached to E which admit a nowhere vanishing continuous vector field $v : E \to T\mathbb{C}^n|_E/TE$ such that $Jv(p)$ points into $\mathcal{W}_1$ and $-Jv(p)$ into $\mathcal{W}_2$, for every $p \in E$.

In the case where $E = \mathbb{R}^n$, the classical edge of the wedge theorem states that there exists a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^n$ such that every function which is continuous on $\mathcal{W}_1 \cup E \cup \mathcal{W}_2$ and holomorphic in $\mathcal{W}_1 \cup \mathcal{W}_2$ extends holomorphically to $\mathcal{D}$. Also, generalizations are known in the case where $f|_{\mathcal{W}_1}$ and $f|_{\mathcal{W}_2}$ have coinciding distributional boundary values on E.

The assumption about the matching up of boundary values along E from $\mathcal{W}_1$ and from $\mathcal{W}_1$ is really needed, even if the two boundary values coincide on a thick subset of E. To support this observation, consider the following elementary example: the complex hyperplane $H := \{z_n = 0\} \subset \mathbb{C}^n$ and the maximally real plane $E := \mathbb{R}^n \subset \mathbb{C}^n$ intersect transversally in the $(n-1)$-dimensional totally real plane $C := \{y = 0, \ x_n = 0\}$; the pair of wedges $\mathcal{W}_1 := \{y_1 > 0, \dots, y_n > 0\}$ and $\mathcal{W}_2 := \{y_1 < 0, \dots, y_n < 0\}$ clearly form a double wedge attached to E; the function $\exp(-1/z_n)$ restricted to the two wedges is holomorphic there, has coinciding boundary values on the thick set $E \backslash C$, but does not extend holomorphically to a neighborhood of E in $\mathbb{C}^n$. Evidently, the envelope of holomorphy of the union of $\mathcal{W}_1 \cup \mathcal{W}_2$ together with a thin neighborhood of $E \backslash C$ in $\mathbb{C}^n$ does *not* contain any neighborhood of E in $\mathbb{C}^n$.

Thus, in order to apply our removability theorems (which are essentially statements about envelopes of holomorphy), the first question is how to impose coincidence of

boundary values on the edge. We shall first see that the fact that $C \subset E$ is exactly of codimension one in the above example is the limiting case for the obstruction to holomorphic extension. Since we want to treat also meromorphic extension, let us remind some definitions.

13.2. Meromorphic functions and envelopes. Let U be a domain in $\mathbb{C}^n$. A meromorphic function $f \in \mathcal{M}(U)$ is a collection of equivalence classes of quotients of locally defined holomorphic functions. It defines a $P_1(\mathbb{C})$-valued function, which is single-valued only on some Zariski dense open subset $D_f \subset U$. More geometrically, we may represent f by the closure Γ_f of its graph $f|_{D_f}$ over D_f, which always constitutes an irreducible n-dimensional complex analytic subset of $U \times P_1(\mathbb{C})$ with surjective, almost everywhere biholomorphic projection onto U (equivalent definition). It is well known that the *indeterminacy set* of f, namely the set of $z \in U$ over which the whole fiber $\{z\} \times P_1(\mathbb{C})$ is contained in Γ_f, is an analytic subset of U of codimension at least 2. It is the only set where f is multivalued.

We shall constantly apply a theorem due to P. Thullen (generalized by S. Ivashkovitch in [I] in the context of Kähler manifolds) according to which the envelope of holomorphy of a domain in $\mathbb{C}^n$ coincides with its envelope of meromorphy. As holomorphic functions are meromorphic, we shall state Lemma 13.4, Corollary 13.8 and Corollary 13.11 below directly for meromorphic functions.

13.3. Edge of the wedge theorem over a maximally real edge. Let $E \subset \mathbb{C}^n$ $(n \geq 2)$ be a real analytic maximally real submanifold, let $(\mathcal{W}_1, \mathcal{W}_2)$ be a double wedge attached to E and let f_1, f_2 be two meromorphic functions in $\mathcal{W}_1$, $\mathcal{W}_2$.

We need an assumption which tames the behaviour of their indeterminacy sets, as one tends towards the edge E from either $\mathcal{W}_1$ or $\mathcal{W}_2$. It will be sufficient to impose a matching up of their boundary values on the complement of a closed subset C whose $(n-1)$-dimensional Hausdorff measure vanishes. Let H_d denote the d-dimensional Hausdorff measure.

Lemma 13.4. *If there is a closed subset $C \subset E$ with $\mathsf{H}_{n-1}(C) = 0$ such that both $\overline{\Gamma_{f_1}} \cap [(E\backslash C) \times P_1(\mathbb{C})]$ and $\overline{\Gamma_{f_2}} \cap [(E\backslash C) \times P_1(\mathbb{C})]$ coincide with the graph of a (single) continuous mapping from $E\backslash C$ to $P_1(\mathbb{C})$, then there exists a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^n$ which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ and a meromorphic function*

$$f \in \mathcal{M}\left(\mathcal{D} \cup \mathcal{W}_1 \cup \mathcal{W}_2\right), \tag{13.5}$$

extending the f_j, namely such that $f|_{\mathcal{W}_j} = f_j$, for $j = 1, 2$.

Proof. First of all, the assumption of continuous coincidence of boundary values enables us to apply the classical edge of the wedge theorem at each point of $E\backslash C$. This yields a neighborhood $\mathcal{D}_0$ of $E\backslash C$ in $\mathbb{C}^n$ and a meromorphic extension $f_0 \in \mathcal{M}(\mathcal{D}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2)$. We claim that the envelope of meromorphy of $\mathcal{D}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ contains a neighborhood $\mathcal{D}_1$ of E in $\mathbb{C}^n$.

Indeed, this follows from a very elementary application of the continuity principle. Let $p \in C$ be arbitrary. After a local straightening, we may insure that p is the origin, that $E = \mathbb{R}^n$, that $\mathcal{W}_1$ contains $\{y_1 > 0, \ldots, y_n > 0\}$ and that $\mathcal{W}_2$ contains $\{y_1 < 0, \ldots, y_n < 0\}$.

Let us introduce the trivial family of analytic discs

$$A_{c,x,v}(\zeta) := (x_1 + c(1+v_1)\zeta, \ldots, x_n + (1+v_n)\zeta)\,, \tag{13.6}$$

where $c > 0$ is a sufficiently small fixed scaling factor, where $x \in \mathbb{R}^n$ is a small translation parameter and where $v \in \mathbb{R}^n$ is a small pivoting parameter. Clearly, $A_{c,x,v}(\partial^+ \Delta)$ is contained in $\mathcal{W}_1$ and $A_{c,x,v}(\partial^- \Delta)$ is contained in $\mathcal{W}_2$. However $A_{c,x,v}(\pm 1)$ may encounter C.

First of all, using the submersiveness of the two mappings $v \mapsto A_{c,0,v}(\pm 1) \in E$, we may find v_0 arbitrarily close to the origin in $\mathbb{R}^n$ such that $A_{c,0,v_0}(\pm 1)$ does not belong to C. It follows that for all small translation vectors $q \in \mathbb{C}^n$, the disc boundary $A_{c,0,v_0}(\partial\Delta) + q$ is contained in the domain $\mathcal{D}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$.

Furthermore, because C is of Hausdorff $(n-1)$-dimensional measure zero, for almost all $x \in \mathbb{R}^n$, the segment $A_{c,x,v_0}([-1,1])$ does not meet C. It follows that for such x, the disc $A_{c,x,v_0}(\overline{\Delta})$ is contained in the domain $\mathcal{D}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$. We deduce that every disc $A_{c,0,v_0}(\partial\Delta) + q$ is analytically isotopic to a point in $\mathcal{D}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$. An application of the continuity principle yields meromorphic extension to a neighborhood of $p = A_{c,0,0}(0) \in C$.

In sum, we have constructed a neighborhood $\mathcal{D}_1$ of E in $\mathbb{C}^n$ and a meromorphic extension $f \in \mathcal{M}(\mathcal{D}_1)$. But $\mathcal{D}_1$ is not independent of (f_1, f_2), since it depends on C. Fortunately, once we know meromorphic extension to a neighborhood $\mathcal{D}_1$ of E in $\mathbb{C}^n$, we may reemploy the analytic disc technique of the classical edge of the wedge theorem to describe a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^n$ which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ (*see* the end of the proof of Corollary 13.8 below for more arguments). This completes the proof of Lemma 13.4. $\square$

13.7. Edge of the wedge theorem over an edge of positive CR dimension. Let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ of positive CR dimension and let C be a proper closed subset of M such that M and $M \backslash C$ are globally minimal. In [MP3], Theorem 1.1, it was shown as a main theorem that every such closed subset C of M is CR-, $\mathcal{W}$- and L^{p}-removable. We may formulate the following application, where, for simplicity, we assume local minimality at every point.

Corollary 13.8. *Let $E \subset \mathbb{C}^n$ $(n \geq 2)$ be a generic manifold of class $\mathcal{C}^{2,\alpha}$ of positive CR dimension which is locally minimal at every point, let $(\mathcal{W}_1, \mathcal{W}_2)$ a double wedge attached to E and let two meromorphic functions $f_j \in \mathcal{M}(\mathcal{W}_j)$ for $j = 1, 2$. If there is a closed subset $C \subset E$ with $\mathsf{H}_{n-1}(C) = 0$ such that both $\overline{\Gamma_{f_1}} \cap [(E \backslash C) \times P_1(\mathbb{C})]$ and $\overline{\Gamma_{f_2}} \cap [(E \backslash C) \times P_1(\mathbb{C})]$ coincide with the graph of a* (*single*) *continuous mapping from $E \backslash C$ to $P_1(\mathbb{C})$, then there exists a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^n$ which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ and a meromorphic function*

$$f \in \mathcal{M}\left(\mathcal{W}_1 \cup \mathcal{D} \cup \mathcal{W}_2\right), \tag{13.9}$$

extending the f_j, namely such that $f|_{\mathcal{W}_j} = f_j$, for $j = 1, 2$.

Proof. Applying the classical edge of the wedge theorem, we get a meromorphic extension $f_0 \in \mathcal{M}(\mathcal{W}_1 \cup \mathcal{D}_0 \cup \mathcal{W}_2)$, where $\mathcal{D}_0$ is some open neighborhood of $E \backslash C$ in $\mathbb{C}^n$. Next, we include E in a CR manifold M with $M \subset \mathcal{W}_1 \cup E \cup \mathcal{W}_2$ and $\dim_{\mathbb{R}} M = 1 + \dim_{\mathbb{R}} E$, as shown in the following figure.

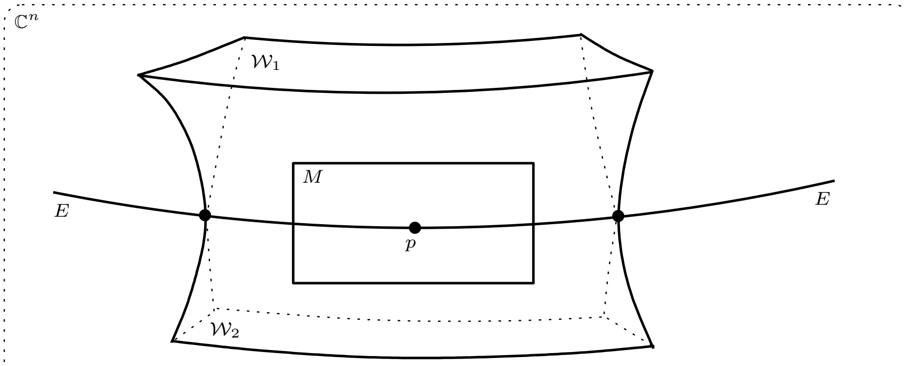

FIGURE 24: A DOUBLE WEDGE $(\mathcal{W}_1, \mathcal{W}_2)$, ITS EDGE E AND THE MANIFOLD M

Of course, the domain $\mathcal{W}_1 \cup \mathcal{D}_0 \cup \mathcal{W}_2$ constitues a (rather thick) wedge attached to $M \backslash C$. Since E is locally minimal at every point and since one CR tangential direction of M is transversal to E, both M and $M \backslash \Sigma$ are both globally minimal. Applying Theorem 1.1 in [MP2], we deduce that there exists a wedge $\mathcal{W}$ attached to M which is contained in the envelope of meromorphy of $\mathcal{W}_1 \cup \mathcal{D}_0 \cup \mathcal{W}_2$. We then claim that there exists a neighborhood $\mathcal{D}$ of p in $\mathbb{C}^n$, which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ such that $\mathcal{D}$ is contained in the envelope of meromorphy of $\mathcal{W}_1 \cup \mathcal{W} \cup \mathcal{W}_2$.

Indeed, by deforming slightly M inside $\mathcal{W}$ near E, we get a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold $M^d \subset \mathcal{W}_1 \cup \mathcal{W} \cup \mathcal{W}_2$. Instead of functions meromorphic in the disconnected open set $\mathcal{W}_1 \cup \mathcal{W}_2$, we now consider meromorphic functions in the *connected* open set $\mathcal{W}_1 \cup \mathcal{W} \cup \mathcal{W}_2$, which is a neighborhood of M^d in $\mathbb{C}^n$. Then by following the proof of the edge of the wedge theorem given in [A] and applying the continuity principle, one deduces meromorphic extension to a neighborhood $\mathcal{D}^d$ in $\mathbb{C}^n$ of the deformed submanifold M^d. Since the size of $\mathcal{W}_1$ and the size of $\mathcal{W}_2$ are uniform with respect to d, the size of the domain $\mathcal{D}^d$ is also uniform with respect to d, as follows from the stability of the edge of the wedge theorem established in [A], since it relies on E. Bishop's equation. Hence for M^d sufficiently close to M, the domain $\mathcal{D}^d$ contains a neighborhood of p in $\mathbb{C}^n$. This completes the proof of Corollary 13.8. □

13.10. Hartogs-Bochner phenomenon and edge of the wedge theorem. Next we turn to the question whether a Hartogs-Bochner phenomenon holds in presence of a double wedge. More precisely we ask when it is sufficient to require coincidence of boundary values only outside some compact $K \subset E$. Let us first look at a prototypical case where the answer is particularly neat, thanks to Theorem 1.1. Obviously, the proof is totally similar to the proof of Corollary 13.8 and will not be repeated.

Corollary 13.11. *Let $E \subset \mathbb{C}^2$ be an embedded real analytic totally real disc, let $(\mathcal{W}_1, \mathcal{W}_2)$ a double wedge attached to E and let two meromorphic functions $f_j \in \mathcal{M}(\mathcal{W}_j)$ for $j = 1, 2$. If there is a compact subset $K \subset E$ such that both $\overline{\Gamma_{f_1}} \cap [(E \backslash K) \times P_1(\mathbb{C})]$ and $\overline{\Gamma_{f_2}} \cap [(E \backslash K) \times P_1(\mathbb{C})]$ coincide with the graph of a (single) continuous mapping from $E \backslash K$ to $P_1(\mathbb{C})$, then there exists a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^2$ which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ and a meromorphic function*

$$f \in \mathcal{M}\left(\mathcal{W}_1 \cup \mathcal{D} \cup \mathcal{W}_2\right), \tag{13.12}$$

extending the f_j, namely such that $f|_{\mathcal{W}_j} = f_j$, for $j = 1, 2$.

In order to find the most general application of Theorems 1.2 and 1.2', we first remark that in FIGURE 24, we have a considerable freedom in the choice of the generic submanifolds M of CR dimension 1 with $E \subset M \subset \mathcal{W}_1 \cup E \cup \mathcal{W}_2$, depending on the aperture of $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\dim_{\mathbb{R}} M = 1 + \dim_{\mathbb{R}} E$, the tangent space to such an M at a point $p \in E$ is uniquely determined by some nonzero vector $v_p \in T_p\mathbb{C}^n / T_pE$. In order that M is locally contained in $\mathcal{W}_1 \cup E \cup \mathcal{W}_2$, it is necessary and sufficient that either Jv points in $\mathcal{W}_1$ and $-Jv$ points in $\mathcal{W}_2$, or vice versa, depending on the orientations of $\mathcal{W}_1$ and $\mathcal{W}_2$ with respect to E. Without loss of generality, after a possible shrinking, we can therefore assume that the cones of $\mathcal{W}_1$ and of $\mathcal{W}_2$ are exactly opposite to each other at every point of E; indeed, it would be impossible to construct an M locally contained in $\mathcal{W}_1 \cup E \cup \mathcal{W}_2$ which satisfies $T_pM = T_pE \oplus \mathbb{R}v_p$ at a point $p \in E$, in the case where the vector Jv_p points in the cone at p of $\mathcal{W}_1$ but $-Jv_p$ lies outside the cone at p of $\mathcal{W}_2$, or vice versa.

Assuming $\mathcal{W}_2$ to be opposite to $\mathcal{W}_1$, let us define an induced field of open cones $p \mapsto \mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$ as follows: a nonzero vector $v_p \in T_pE \backslash \{0\}$ belongs to $\mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$ if either Jv or $-Jv$ points into $\mathcal{W}_1$. A nowhere vanishing vector field $p \mapsto v(p)$ is said to be *directed by* $\mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$ if $v(p) \in \mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$ for every $p \in E$. Clearly, for every vector field $p \mapsto v(p)$ directed by $p \mapsto \mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$, we may construct a $\mathcal{C}^{2,\alpha}$-smooth semi-local generic submanifold M containing E, contained in $\mathcal{W}_1 \cup E \cup \mathcal{W}_2$ which satisfies $T_pM = T_pE \oplus Jv(p)$ at every point $p \in E$.

In the statement of Theorem 1.2', we defined the condition $\mathcal{F}_{M^1}^c\{C\}$ with respect to some CR manifold M containing the totally real manifold M^1. But M entered in the definition only via the characteristic foliation induced on M^1. Hence its reasonable to define a more general nontransversality property, by replacing the characteristic foliation by the foliation induced by any vector field directed by the field of cones $p \mapsto \mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$, as follows:

$\mathcal{F}_{\mathcal{W}_1,\mathcal{W}_2}\{C\}$: For every closed subset $C' \subset C$ there is a smooth vector field $p \mapsto v(p)$ directed by the field of cones $p \mapsto \mathsf{C}_p^{\mathcal{W}_1,\mathcal{W}_2}$ such that there exists a simple $\mathcal{C}^{2,\alpha}$-smooth curve $\gamma' : [-1,1] \to E$ whose range $\gamma'([-1,1])$ is contained in a single integral curve of $p \mapsto v(p)$ with $\gamma'(-1) \notin C'$, $\gamma'(0) \in C'$ and $\gamma'(1) \notin C'$, there exists a local $(n-1)$-dimensional transversal $R \subset E$ to γ' passing through $\gamma'(0)$ and there exists a thin open neighborhood V of $\gamma'([-1,1])$ in E such that if $\pi : V \to R$ denotes the semi-local projection parallel to the flow lines of v, then $\gamma'(0)$ lies on the boundary, relatively to the topology of R, of $\pi(C' \cap V)$.

The proper application of Theorem 1.2' is the following. Its proof follows by a direct examination of the proof of Theorem 1.2'.

Corollary 13.13. *Let $E \subset \mathbb{C}^n$ $(n \geq 2)$ be a real analytic maximally real submanifold, let $(\mathcal{W}_1, \mathcal{W}_2)$ a double wedge attached to E. Let C be a proper closed subset of E satisfying the nontransversality property $\mathcal{F}_{\mathcal{W}_1,\mathcal{W}_2}\{C\}$ above. Let two meromorphic functions $f_j \in \mathcal{M}(\mathcal{W}_j)$ for $j = 1, 2$ such that both $\overline{\Gamma_{f_1}} \cap [(E\backslash C) \times P_1(\mathbb{C})]$ and $\overline{\Gamma_{f_2}} \cap [(E\backslash C) \times P_1(\mathbb{C})]$ coincide with the graph of a (single) continuous mapping from $E\backslash C$ to $P_1(\mathbb{C})$, Then there exists a neighborhood $\mathcal{D}$ of E in $\mathbb{C}^2$ which depends only on $(\mathcal{W}_1, \mathcal{W}_2)$ and a meromorphic function*

$$f \in \mathcal{M}(\mathcal{W}_1 \cup \mathcal{D} \cup \mathcal{W}_2), \tag{13.14}$$

extending the f_j, namely such that $f|_{\mathcal{W}_j} = f_j$, for $j = 1, 2$.

13.15. Further applications. We now conclude this section by suggesting two applications of Theorem 1.2' in higher dimensions, in the case where M^1 is not everywhere

totally real. However, we must mention that we did not try to generalize the results of E. Bishop to understand the local geometry of complex tangencies of generic submanifolds of CR dimension 1 in $\mathbb{C}^n$, for $n \geq 3$. Consequently, our formulations should be considered as mild generalizations of Theorem 1.1 and 1.3.

Thus, let M be a $\mathcal{C}^{2,\alpha}$-smooth generic submanifold of $\mathbb{C}^n$ ($n \geq 2$) of CR dimension 1, let M^1 be a codimension one submanifold of M which is maximally real except at every point of some proper closed subset $E \subset M^1$. Let C be a proper closed subset of M^1. For simplicity, we assume that M is locally minimal at every point, an assumption which insures that for every closed subset $\widetilde{C}$ of M, both M and $M \backslash \widetilde{C}$ are globally minimal.

Firstly, applying Theorem 1.2' to remove $C \cap (M^1 \backslash E)$ and then Theorem 1.1 of [MP2] to remove $C \cap E$, we deduce the following.

Corollary 13.16. *Assume that E is of vanishing $(n-1)$-dimensional Hausdorff content, and that the nontransversality condition $\mathcal{F}^c_{M^1 \backslash E}\{C \cap (M^1 \backslash E)\}$ holds. Then C is CR-, $\mathcal{W}$- and L^{p}-removable.*

Secondly, we may generalize the notion of hyperbolic tree and assume that E consists of finitely many compact submanifolds of codimension 2 in M^1 joined by a collection of finitely many codimension one submanifolds of M^1 with boundaries in E which are foliated by characteristic curves. Under some easily found assumptions, one could formulate a second corollary analogous to Theorem 1.3.

§14. An example of a nonremovable three-dimensional torus

This final section is devoted to exhibit a crucial example of a closed subset C violating the main nontransversality condition $\mathcal{F}^c_{M^1}\{C\}$ of Theorem 1.2' such that C is truly nonremovable. In addition, we may require that M and M^1 have the simplest possible topology.

Lemma 14.1. *There exists a triple (M, M^1, C), where*

- **(i)** *M is a $\mathcal{C}^\infty$-smooth generic submanifold in $\mathbb{C}^3$ of CR dimension 1, diffeomorphic to a real 4-ball*;
- **(ii)** *M^1 is a $\mathcal{C}^\infty$-smooth one-codimensional submanifold of M which is maximally real in $\mathbb{C}^n$ and diffeomorphic to a real 3-ball*;
- **(iii)** *C is a compact subset of M^1 diffeomorphic to a real three-dimensional torus which is everywhere transversal to the characteristic foliation $\mathcal{F}^c_{M^1}$, hence the nontransversality condition $\mathcal{F}^c_{M^1}\{C\}$ of Theorem 1.2' clearly does not hold*;
- **(iv)** *M of finite type 4 in the sense of T. Bloom and I. Graham at every point, hence globally minimal,*

such that C is neither CR- nor $\mathcal{W}$- nor L^{p}-removable with respect to M.

By type 4 at a point $p \in M$, we mean of course that the Lie brackets of the complex tangent bundle T^cM up to length 4 generate T_pM.

14.2. The geometric recipe. We first construct the 3-torus C, then construct the maximally real M^1 and finally define M as a certain thickening of M^1. The argument for insuring global minimality of M involves computations with Lie brackets and is postponed to the end.

Firstly, in $\mathbb{R}^3 = \mathbb{R}^3 \oplus i\{0\} \subset \mathbb{C}^3$ equipped with the coordinates (x_1, x_2, x_3), where $x_j = \operatorname{Re} z_j$ for $j = 1, 2, 3$, pick the "standard" 2-dimensional torus T^2 of Cartesian

equation

$$\left(\sqrt{x_1^2+x_2^2}-2\right)^2+x_3^2=1. \tag{14.3}$$

This torus is stable under the rotations directed by the x_3-axis; its intersection with the (x_1, x_3)-plane consists of two circles of radius 1 centered at the points $x_1 = 2$ and $x_1 = -2$; it bounds a three-dimensional open "full" torus T^3; both T^2 and T^3 are contained in the ball B^3 of radius 5 centered at the origin.

It is better to drop the square root: one checks that the equations of T^2 and T^3 are equally given by $T^2 := \{\rho = 0\}$ and $T^3 := \{\rho < 0\}$, by means of the *polynomial* defining function

$$\rho(x_1,x_2,x_3) := (x_1^2+x_2^2+x_3^2+3)^2-16\,(x_1^2+x_2^2), \tag{14.4}$$

which has nonvanishing differential at every point of T^2. Consequently, the extrinsinc complexification of T^2, namely the complex hypersurface defined by

$$\Sigma := \{(z_1,z_2,z_3)\in\mathbb{C}^3 :\ \rho(z_1,z_2,z_3)=0\} \tag{14.5}$$

cuts $\mathbb{R}^3$ along T^2 with the transversality property $T_x\mathbb{R}^3\cap T_x\Sigma = T_xT^2$ for every point $x\in T^2$.

Secondly, according to G. Reeb (*see* [CLN], pp. 25–27; *see* also the figures there), by considering the space $\mathbb{R}^3\equiv S^3\backslash\{\infty\}$ as a punctured three-dimensional sphere S^3, one may glue a second three-dimensional full torus $\widetilde{T}^3$ to T^3 along T^2 with $\infty\in\widetilde{T}^3$ and then construct a foliation of S^3 by 2-dimensional surfaces all of whose leaves, except one, are diffeomorphic to $\mathbb{R}^2$, are contained in either T^3 or in $\widetilde{T}^3$ and are accumulating on T^2, and finally, whose single compact leaf is the above 2-torus T^2. This yields the so-called *Reeb foliation* of S^3, which is $\mathcal{C}^\infty$-smooth and orientable. Consequently, there exists a $\mathcal{C}^\infty$ smooth vector field $L = a_1(x)\,\partial_{x_1}+a_2(x)\,\partial_{x_2}+a_3(x)\,\partial_{x_3}$ of norm 1, namely $a_1(x)^2+a_2(x)^2+a_3(x)^2=1$ for every $x\in\mathbb{R}^3$, which is everywhere orthogonal (with respect to the standard Euclidean structure) to the leaves of the Reeb foliation. Geometrically, the integral curves of L accumulate asymptotically on the two nodal (central) circles of T^3 and of $\widetilde{T}^3$.

The open ball $B^3\subset\mathbb{R}^3$ of radius 5 centered at the origin will be our maximally real submanifold M^1. The two-dimensionaly torus T^2 will be our nonremovable closed set C. The integral curves of the vector field L will be our characteristic lines. Since L is orthogonal to T^2, these characteristic lines will of course be everywhere transverse to C, so that $\mathcal{F}^c_{M^1}\{C\}$ clearly does not hold.

Thirdly, it remains to construct the generic submanifold M of CR dimension 1 containing M^1 and to check that C will be nonremovable.

First of all, we notice that L provides the characteristic directions of M^1 if and only if $T_xM = T_p\mathbb{R}^3\oplus\mathbb{R}\,J\,L(x)$ for every point $x\in M^1\equiv B^3$. Consequently, all submanifolds $M\subset\mathbb{C}^3$ obtained by slightly thickening M^1 in the direction of $J\,L(x)$ will be convenient; in other words, only the first jet of M along M^1 is prescribed by our choice of the characterisctic vector field L. Notice that all such thin strips M along M^1 will be diffeomorphic to a real 4-ball.

The fact that C is nonremovable for all such generic submanifolds M is now clear: the hypersurface $\Sigma = \{z\in\mathbb{C}^3 :\ \rho(z)=0\}$ satisfying $T_x\Sigma = T_xT^2\oplus\mathbb{R}\,J\,T_xT^2$ for all $x\in T^2$ and L being transversal to T^2, we easily deduce the transversality property $T_x\Sigma + T_xM = T_x\mathbb{C}^3$ for all $x\in T^2$, a geometric property which insures that the holomorphic function $1/\rho(z)$, which is CR on $M\backslash C$, does not extend holomorphically to any

wedge of edge M at any point of C. Intuitively, $T_x\Sigma/T_xM$ absorbs all the normal space $T_x\mathbb{C}^3/T_xM$ at every point $x \in T^2$, leaving no room for any open cone.

Finally, to fulfill all the hypotheses of Theorem 1.2' (except of course $\mathcal{F}_{M^1}^c\{C\}$), we have to insure that M is globally minimal. We claim that by bending strongly the second and the fourth order jet of M along M^1 (without modifying the first order jet which must be prescribed by $J\,L$), one may insure that M is of type 4 in the sense of T. Bloom and I. Graham at every point of M^1; since being of finite type is an open property, it follows that M is finite type at every point provided that, as a strip, M is sufficiently thin along M^1. As is known, finite-typeness at every point implies local minimality at every point which in turn implies global minimality. This completes the recipe.

We would like to mention that by following a similar recipe, one may construct an elementary example of a non-removable compact subset of a generic submanifold of codimension one diffeomorphic to a 4-ball lying in a globally minimal hypersurface in $\mathbb{C}^3$ which is (also) diffeomorphic to a 5-ball (*cf.* [JS]).

14.6. Finite-typisation. Thus, it remains to construct a generic submanifold $M \subset \mathbb{C}^3$ of CR dimension 1 satisfying $T_xM = T_xM^1 \oplus \mathbb{R}\,J\,L(x)$ for every $x \in M^1$, which is of *type* 4 *at every point* $x \in M^1$.

First of all, let us denote by $L = a_1(x)\,\partial_{x_1} + a_2(x)\,\partial_{x_2} + a_3(x)\,\partial_{x_3}$ the unit vector field which was constructed as a field orthogonal to the Reeb foliation: it is defined over $\mathbb{R}^3$ and has $\mathcal{C}^\infty$-smooth coefficients satisfying $a_1(x)^2 + a_2(x)^2 + a_3(x)^2 = 1$ for all $x \in \mathbb{R}^2$. The two-dimensional quotient vector bundle $T\mathbb{R}^3/(\mathbb{R}L)$ with contractible base being necessarily trivial, it follows that we can complete L by two other $\mathcal{C}^\infty$-smooth unit vector fields K^1 and K^2 defined over $\mathbb{R}^3$ such that the triple $(L(x), K^1(x), K^2(x))$ forms a direct orthonormal frame at every point $x \in \mathbb{R}^3$. Let us denote the coefficients of K^1 and of K^2 by

$$\begin{aligned} K^1 &= \rho_1\,\partial_{x_1} + \rho_2\,\partial_{x_2} + \rho_3\,\partial_{x_3}, \\ K^2 &= r_1\,\partial_{x_1} + r_2\,\partial_{x_2} + r_3\,\partial_{x_3}, \end{aligned} \tag{14.7}$$

where ρ_j and r_j for $j = 1, 2, 3$ are $\mathcal{C}^\infty$-smooth functions of $x \in \mathbb{R}^3$ satisfying $\rho_1^2 + \rho_2^2 + \rho_3^2 = 1$ and $r_1^2 + r_2^2 + r_3^2 = 1$. In our case, K^1 and K^2 may even be constructed directly by means of a trivialization of the bundle tangent to the Reeb foliation.

Let $P > 0$ be a constant, which will be chosen later to be large. Since by construction we have the two orthogonality relations $a_1\rho_1 + a_2\rho_2 + a_3\rho_3 = 0$ and $a_1r_1 + a_2r_2 + a_3r_3 = 0$, it follows that every generic submanifold $M_P \subset \mathbb{C}^3$ defined by the two Cartesian equations

$$\begin{aligned} 0 &= \rho = y_1\,\rho_1(x) + y_2\,\rho_2(x) + y_3\,\rho_3(x) + P\left[y_1^2 + y_2^2 + y_3^2\right], \\ 0 &= r = y_1\,r_1(x) + y_2\,r_2(x) + y_3\,r_3(x) + P^3\left[y_1^4 + y_2^4 + y_3^4\right] \end{aligned} \tag{14.8}$$

enjoys the property that the vector field $JL(x) = a_1(x)\,\partial_{y_1} + a_2(x)\,\partial_{x_2} + a_3(x)\,\partial_{x_3}$ is tangent to M_P at every $x \in \mathbb{R}^3$. As desired, we deduce that $T_x^cM = \mathbb{R}L(x) \oplus J\mathbb{R}L(x)$ for every $x \in \mathbb{R}^3$, a property which insures that $\mathbb{R}L(x)$ is the characteristic direction of M^1 in M_P, independently of P.

To complete the final minimalization argument for the construction of a nonremovable compact set $C := T^2 \subset M^1 \subset M$ which appears in the Introduction, it suffices now to apply the following lemma with $R = 5$. Though calculatory, its proof is totally elementary.

Lemma 14.9. *For every $R > 0$, there exist $P > 0$ sufficiently large such that M_P is of type 4 at every point $x \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leq R^2$.*

Proof. As above, let $M_P = \{z \in \mathbb{C}^3 : \rho = r = 0\}$. By writing the tangency condition, one checks immediately that the one-dimensional complex vector bundle $T^{1,0}M_P$ is generated over $\mathbb{C}$ by the vector field $\mathbb{L} := A_1\,\partial_{z_1} + A_2\,\partial_{z_2} + A_3\,\partial_{z_3}$, with the explicit expressions

$$\begin{aligned} A_1 &:= 4\rho_{z_3} r_{z_2} - 4\rho_{z_2} r_{z_3}, \\ A_2 &:= 4\rho_{z_1} r_{z_3} - 4\rho_{z_3} r_{z_1}, \\ A_3 &:= 4\rho_{z_2} r_{z_1} - 4\rho_{z_1} r_{z_2}. \end{aligned} \tag{14.10}$$

Using the expressions (14.8) for ρ and r, we see that these three components restrict on $\{y = 0\}$ as the Plücker coordinates of the bivector (K^1, K^2), namely

$$\begin{aligned} A_1|_{y=0} &= \rho_2 r_3 - \rho_3 r_2 =: \Delta_{2,3}, \\ A_2|_{y=0} &= \rho_3 r_1 - \rho_1 r_3 =: \Delta_{3,1}, \\ A_3|_{y=0} &= \rho_1 r_2 - \rho_2 r_1 =: \Delta_{1,2}. \end{aligned} \tag{14.11}$$

As K^1 and K^2 are of norm 1 and orthogonal at every point, it follows by direct computation that $\Delta_{2,3}^2 + \Delta_{3,1}^2 + \Delta_{1,2}^2 = 1$ and that the vector of coordinates $(\Delta_{2,3}, \Delta_{3,1}, \Delta_{1,2})$ is orthogonal to both K^1 and K^2. Moreover, as the orthonormal trihedron $(L(x), K^1(x), K^2(x))$ is direct at every point, we deduce that necessarily

$$\Delta_{2,3} \equiv a_1, \qquad \Delta_{3,1} \equiv a_2, \qquad \Delta_{1,2} \equiv a_3. \tag{14.12}$$

Next, we compute in length A_1, A_2 and A_3 using (14.8). As their complete explicit development will not be crucial for the sequel and as we shall perform with them differentiations and linear combinations yielding relatively complicated expressions, let us adopt the following notation: by $\mathcal{R}^0$, we denote various expressions which are polynomials in the jets of the functions ρ_1, ρ_2, ρ_3 and r_1, r_2, r_3. Similarly, by $\mathcal{R}^I$, by $\mathcal{R}^{II}$, by $\mathcal{R}^{III}$ and by $\mathcal{R}^{IV}$, we denote polynomials in the transverse variables (y_1, y_2, y_3) which are homogeneous of degree 1, 2, 3 and 4 and have as coefficients various expressions $\mathcal{R}^0$.

Importantly, we make the convention that such expressions $\mathcal{R}^0$, $\mathcal{R}^I$, $\mathcal{R}^{II}$, $\mathcal{R}^{III}$ and $\mathcal{R}^{IV}$ should be totally independent of the constant P. Consequently, if P appears somehow, we shall write it as a factor, as for instance in $P\,\mathcal{R}^I$ or in $P^3\,\mathcal{R}^{III}$.

With this convention at hand, we may develope (14.10) using the expressions (14.8) by writing out only the terms which will be useful in the sequel and by treating the rest as controlled remainders. Let us detail the computation of A_1:

(14.13)

$$\begin{aligned} A_1 \ &= 4\left[-\frac{i}{2}\rho_3 - iPy_3 + \mathcal{R}^I\right]\left[-\frac{i}{2}r_2 - 2iP^3y_2^3 + \mathcal{R}^I\right] - \\ &\quad - 4\left[-\frac{i}{2}\rho_2 - iPy_2 + \mathcal{R}^I\right]\left[-\frac{i}{2}r_3 - 2iP^3y_3^3 + \mathcal{R}^I\right] \\ &= -\rho_3 r_2 - 4P^3\rho_3 y_2^3 + \mathcal{R}^I - 2Pr_2y_3 + P^4\mathcal{R}^{IV} + P\mathcal{R}^I + \mathcal{R}^I + P^3\mathcal{R}^{IV} + \mathcal{R}^{II} \\ &\quad + \rho_2 r_3 + 4P^3\rho_2 y_3^3 + \mathcal{R}^I + 2Pr_3y_2 + P^4\mathcal{R}^{IV} + P\mathcal{R}^I + \mathcal{R}^I + P^3\mathcal{R}^{IV} + \mathcal{R}^{II} \\ &= \rho_2 r_3 - \rho_3 r_2 + 2Pr_3y_2 - 2Pr_2y_3 + 4P^3\rho_2 y_3^3 - 4P^3\rho_3 y_2^3 + \\ &\quad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}. \end{aligned}$$

In the development, before simplification, we firstly write out in lines 3 and 4 all the 9×2 terms of the two product: for instance, the third term of the first product, namely

$4(-\frac{i}{2}\rho_3)(\mathcal{R}^I)$, yields a term $\mathcal{R}^I$ whereas the fifth term $4(-iPy_3)(-2iP^3y_2^3)$ yields a term $P^4\mathcal{R}^{IV}$; secondly, we simplify the obtained sum: by our convention, $\mathcal{R}^I + \mathcal{R}^I = \mathcal{R}^I$, whereas $\mathcal{R}^I + P\mathcal{R}^I$ cannot be simplified, since the large constant P will be chosen later. With these technical explanations at hand, we shall not provide any intermediate detail for the further computations, whose rules are totally analogous. For A_1, A_2 and A_3, we obtain

$$\left\{\begin{aligned} A_1 &= \rho_2 r_3 - \rho_3 r_2 + 2Pr_3y_2 - 2Pr_2y_3 + 4P^3\rho_2y_3^3 - 4P^3\rho_3y_2^3 + \\ &\quad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}, \\ A_2 &= \rho_3 r_1 - \rho_1 r_3 + 2Pr_1y_3 - 2Pr_3y_1 + 4P^3\rho_3y_1^3 - 4P^3\rho_1y_3^3 + \\ &\quad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}, \\ A_3 &= \rho_1 r_2 - \rho_2 r_1 + 2Pr_2y_1 - 2Pr_1y_2 + 4P^3\rho_1y_2^3 - 4P^3\rho_2y_1^3 + \\ &\quad + \mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^3\mathcal{R}^{IV} + P^4\mathcal{R}^{IV}. \end{aligned}\right. \tag{14.14}$$

Now that we have written the complex vector field $\mathbb{L}$ and its coefficients A_1, A_2 and A_3, in order to establish Lemma 14.9, it suffices to choose $P > 0$ sufficiently large in order that the four complex vector fields

$$\overline{\mathbb{L}}\,|_{y=0}, \quad \mathbb{L}|_{y=0}, \quad \big[\overline{\mathbb{L}}, \mathbb{L}\big]\,|_{y=0}, \quad \big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \mathbb{L}\big]\big]\big]\,|_{y=0} \tag{14.15}$$

are linearly independent at every point $x \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leq R^2$. At the end of the proof, we shall explain why we cannot insure type 3 at every point, namely why the consideration of $\big[\overline{\mathbb{L}}, \big[\overline{\mathbb{L}}, \mathbb{L}\big]\big]\,|_{y=0}$ instead of the length four last Lie bracket in (14.15) would fail.

As promised, we shall now summarize all the subsequent computations. As we aim to restrict the last Lie bracket to $\{y = 0\}$ which is of length four and whose coefficients involve derivatives of order at most three of the coefficients A_1, A_2 and A_3, we can already neglect the last two remainders $P^3\mathcal{R}^{IV}$ and $P^4\mathcal{R}^{IV}$ in (14.14). In other words, we can consider A^1, A^2 and $A^3 \mod (IV)$. Similarly, in the computation of the Lie bracket

$$\big[\overline{\mathbb{L}}, \mathbb{L}\big] =: C_1\,\partial_{z_1} + C_2\,\partial_{z_2} + C_3\,\partial_{z_3} - \overline{C_1}\,\partial_{\bar z_1} - \overline{C_2}\,\partial_{\bar z_2} - \overline{C_3}\,\partial_{\bar z_3}, \tag{14.16}$$

before restriction to $\{y = 0\}$, we can restrict our task to developing the coefficients

$$\begin{aligned} C_1 &:= \overline{A_1}A_{1,\bar z_1} + \overline{A_2}A_{1,\bar z_2} + \overline{A_3}A_{1,\bar z_3}, \\ C_2 &:= \overline{A_1}A_{2,\bar z_1} + \overline{A_2}A_{2,\bar z_2} + \overline{A_3}A_{2,\bar z_3}, \\ C_3 &:= \overline{A_1}A_{3,\bar z_1} + \overline{A_2}A_{3,\bar z_2} + \overline{A_3}A_{3,\bar z_3} \end{aligned} \tag{14.17}$$

only modulo order (III), which yields by means of the expressions (14.14)

$$\begin{aligned} C_1 \mod (III) &\equiv -iP\rho_1 + 6iP^3a_3\rho_2y_3^2 - 6iP^3a_2\rho_3y_2^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}, \\ C_2 \mod (III) &\equiv -iP\rho_2 + 6iP^3a_1\rho_3y_1^2 - 6iP^3a_3\rho_1y_3^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}, \\ C_3 \mod (III) &\equiv -iP\rho_3 + 6iP^3a_2\rho_1y_2^2 - 6iP^3a_1\rho_2y_1^2 + \mathcal{R}^0 + \mathcal{R}^I + \\ &\quad + P\mathcal{R}^I + P^2\mathcal{R}^I + \mathcal{R}^{II} + P\mathcal{R}^{II} + P^2\mathcal{R}^{II}. \end{aligned} \tag{14.18}$$

We must mention the use of natural rule hold for computing the partial derivatives $A_{j,\bar z_k}$: we have for instance $\partial_{\bar z_k}\big(\mathcal{R}^{II}\big) = \mathcal{R}^I + \mathcal{R}^{II}$. Also, we have used the hypothesis that

$(L(x), K^1(x), K^2(x))$ provides a direct orthonormal frame at every $x \in \mathbb{R}^3$, which yields in particular the three relations

$$a_2 r_3 - a_3 r_2 = -\rho_1, \quad a_3 r_1 - a_1 r_3 = -\rho_2, \quad a_1 r_2 - a_2 r_1 = -\rho_3. \tag{14.19}$$

After mild computation, the coefficients F_1, F_2 and F_3 of the length four Lie bracket

$$\left[\overline{\mathbb{L}}, \left[\overline{\mathbb{L}}, \left[\overline{\mathbb{L}}, \mathbb{L}\right]\right]\right] = F_1\, \partial_{z_1} + F_2\, \partial_{z_2} + F_3\, \partial_{z_3} + G_1\, \partial_{\bar{z}_1} + G_2\, \partial_{\bar{z}_2} + G_3\, \partial_{\bar{z}_3} \tag{14.20}$$

are given, after restriction to $\{y = 0\}$, by

$$\begin{aligned} F_1|_{y=0} &= 3iP^3 a_2^3 \rho_3 - 3iP^3 a_3^3 \rho_2 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \\ F_2|_{y=0} &= 3iP^3 a_3^3 \rho_1 - 3iP^3 a_1^3 \rho_3 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \\ F_3|_{y=0} &= 3iP^3 a_1^3 \rho_2 - 3iP^3 a_2^3 \rho_1 + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0, \end{aligned} \tag{14.21}$$

We can now complete the proof of Lemma 14.9. In the basis $(\partial_{z_1}, \partial_{z_2}, \partial_{z_3}, \partial_{\bar{z}_1}, \partial_{\bar{z}_2}, \partial_{\bar{z}_3})$, the 4×6 matrix associated with the four vector fields (14.15) (without mentioning $|_{y=0}$)

$$\begin{pmatrix} 0 & 0 & 0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \\ C_1 & C_2 & C_3 & -\overline{C_1} & -\overline{C_2} & -\overline{C_3} \\ F_1 & F_2 & F_3 & G_1 & G_2 & G_3 \end{pmatrix} \tag{14.22}$$

has rank four at a point $x \in \mathbb{R}^3$ if and only if the 3×3 determinant in the left low corner is nonvanishing, namely if and only if the developped expression

(14.23)

$$\begin{aligned} &\begin{vmatrix} a_1 & a_2 & a_3 \\ -iP\rho_1 + \mathcal{R}^0 & -iP\rho_2 + \mathcal{R}^0 & -iP\rho_3 + \mathcal{R}^0 \\ \begin{array}{c} 3iP^3a_2^3\rho_3 - 3iP^3a_3^3\rho_2 + \\ +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 \end{array} & \begin{array}{c} 3iP^3a_3^3\rho_1 - 3iP^3a_1^3\rho_3 + \\ +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 \end{array} & \begin{array}{c} 3iP^3a_1^3\rho_2 - 3iP^3a_2^3\rho_1 + \\ +\mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 \end{array} \end{vmatrix} \\ &= 3P^4 \left(r_3[a_1^3\rho_2 - a_2^3\rho_1] + r_2[a_3^3\rho_1 - a_1^3\rho_3] + r_1[a_2^3\rho_3 - a_3^3\rho_2]\right) + \\ &\quad + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 + P^3\mathcal{R}^0 + P^4\mathcal{R}^0 \\ &= 3P^4 \left(a_1^4 + a_2^4 + a_3^4\right) + \mathcal{R}^0 + P\mathcal{R}^0 + P^2\mathcal{R}^0 + P^3\mathcal{R}^0 + P^4\mathcal{R}^0 \end{aligned}$$

is nonvanishing.

At this point, the conclusion of the lemma is now an immediate consequence of the following trivial assertion: *Let a_1, a_2 and a_3 be $\mathcal{C}^\infty$-smooth functions on $\mathbb{R}^3$ satisfying $a_1(x)^2 + a_2(x)^2 + a_3(x)^2 = 1$ for all $x \in \mathbb{R}^3$ and let $\mathcal{R}_0^0$, $\mathcal{R}_1^0$, $\mathcal{R}_2^0$, $\mathcal{R}_3^0$ and $\mathcal{R}_4^0$ be $\mathcal{C}^\infty$-smooth functions on $\mathbb{R}^3$. For every $R > 0$, there exists a constant $P > 0$ large enough so that the function*

$$3P^4 \left(a_1^4 + a_2^4 + a_3^4\right) + \mathcal{R}_0^0 + P\mathcal{R}_1^0 + P^2\mathcal{R}_2^0 + P^3\mathcal{R}_3^0 + P^4\mathcal{R}_4^0 \tag{14.24}$$

is positive at every $x \in \mathbb{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leq R^2$.

If we had put $y_1^3 + y_2^3 + y_3^3$ instead of $y_1^4 + y_2^4 + y_3^4$ in the second equation (14.8), we would have considered the length three Lie bracket $\left[\overline{\mathbb{L}}, \left[\overline{\mathbb{L}}, \mathbb{L}\right]\right]|_{y=0}$ instead of the length four Lie bracket in (14.15), and hence instead of the quartic $a_1^4 + a_2^4 + a_3^4$ in (14.24), we would have obtained the cubic $a_1^3 + a_2^3 + a_3^3$, a function which (unfortunately) vanishes, for instance if $a_1(x) = \frac{1}{\sqrt{2}}$, $a_2(x) = -\frac{1}{\sqrt{2}}$ and $a_3(x) = 0$. We notice that in our example, this value of (a_1, a_2, a_3) is indeed attained at the point $x \in T^2$ of coordinates $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 0)$, whence the necessity of passing to type 4. The proof of Lemma 14.9 is complete. $\square$

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